Realizing Trigonometric Functions with the Multi-Function Converter LH0094

National Semiconductor
Application Note 637
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This note discusses how to use the LH0094 to generate output voltages that are the sine and the cosine of the input voltage. Circuits are developed and the elements dimensioned. Special emphasis is given to the procedure.

The LH0094 can be used to generate a wide variety of output voltages as a function of three input voltages. One application is to generate trigonometric functions.

The LH0094's transfer function is dependent on three inputs, Vx, Vy, and Vz:

$$E_o = Vy \times (Vz/Vx)^m, \qquad \text{(eq. 1)}$$

with m between 0.1 and 10. All voltages are positive. Two resistors are used to set the value of m (R1, R2 in *Figure 2*). E_o is the output voltage of the LH0094. In this application Vx and Vy are held constant at 10V.

In order to be realized by the LH0094 the trig function needs to be approximated by a polynomial. This can be done by a Taylor series:

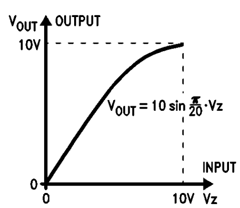
$$f(x) = f(xo) + x \times f'(xo) + (x^2/2) \times f''(xo) + \ldots$$

where xo is the point where the series is developed. This is also the point of highest accuracy and therefore chosen close to the middle of the range of x.

The remainder of the series is truncated, because the LH0094 can provide only one exponential function. For the Taylor series this is the x^2 term. The constant and linear terms are added with a summing amplifier *(Figure 3)*.

A better approximation can be achieved if the highest exponent is made a fraction rather than an integer as is the case for the Taylor series. The LH0094 can accommodate this easily. The values of the coefficients and of the exponent can then be determined in such a way that the error within the value range of x is minimized.

In algebra the independent variable is normally called x. In this application the independent variable is the input voltage Vz, while the dependent variable, normally called y, is the output voltage V_{OUT}.



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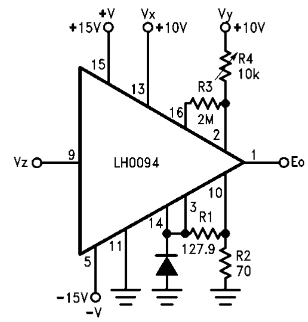
The independent variable has been named Vz to conform with the convention of the LH0094. Both Vy and Vz are voltages.

FIGURE 1. Sinus Function to be Realized



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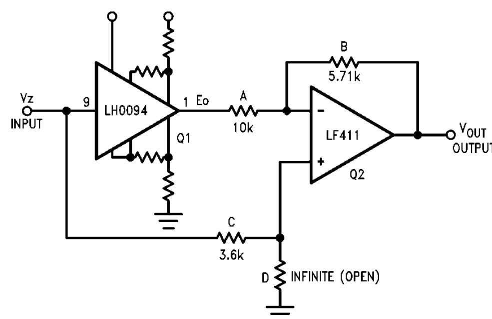
The clamp diode is for protection. The potentiometer R4 is used to trim the output E_o to 10V when all 3 inputs Vx, Vy, and Vz are at +10V.

FIGURE 2. The Multifunction Generator LH0094 Connected to Realize the Function $E_o = 10 \times (Vz/10)^{2.827}$



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Q1 is the LH0094 connected as shown in *Figure 2*. Its output is proportional to the exponential part of the transfer function.

FIGURE 3. Circuit to Realize the Sine Function

I. REALIZING THE FUNCTION y = sin x

1. OBJECTIVE

A circuit needs to be designed where the output voltage is the sine of the input voltage, as shown in *Figure 1*. The basic function is

$$y = \sin x.$$

If substitutions are made: $x = c \times Vz$ and $y = V_{OUT}/k$, the function becomes

$$V_{OUT} = k \times \sin c \times Vz \qquad \text{(eq. 2)}$$

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This equation connects input and output voltage by a sine function. The values of k and c can be determined so that these boundary conditions are met:

at $V_{IN} = Vz = 0V$ ... $V_{OUT} = 0V$ and the argument $c \times Vz = 0$ radians, at $V_{IN} = Vz = 10V$ ... $V_{OUT} = 10V$ and the argument $c \times Vz = \pi/2$ rad.

Both k and 1/c have the dimension of a voltage, and therefore the value of the sine (V_{OUT}/k) and its argument ($c \times Vz$) remain without dimension. In the following equations all voltages need to be measured in Volts.

2. DEVELOPMENT OF A REALIZABLE FUNCTION

From eq. 2 and its boundary conditions it can be shown that

$$c = \frac{1}{10V} \times \frac{\pi}{2} \text{ and } k = 10V$$

The desired function (eq. 2) becomes

$$\frac{V_{OUT}}{10V} = \sin \frac{\pi}{2} \times \frac{V_Z}{10V}, \text{ or, in a shorter form,}$$

$$V_{OUT} = 10 \sin (Vz \times \pi/20), \qquad \text{(eq. 3)}$$

where Vz is the input voltage.

Because the LH0094 can only create an exponential function, the sine function in eq. 3 needs to be substituted by a polynomial. The approximation

$$\sin x = x - x^m/6.28, \qquad \text{(eq. 4)}$$

with $x = Vz \times \pi/20$ and $m = 2.827$

lets the desired function be written as

$$V_{OUT} = 10\,[Vz \times \pi/20 - (Vz \times \pi/20)^m/6.28]. \qquad \text{(eq. 5)}$$

This approximation is accurate for Vz from 0V to +10V with a theoretical accuracy of 0.23%.

3. DEVELOPMENT OF THE CIRCUIT

3a. Realization of the Exponential Term

The first task is to realize a function proportional to Vz^m. Then it can be scaled and added to the other terms to yield the required transfer function (eq. 5).

Figure 2 shows how to connect the LH0094 to perform the exponential portion of the desired function (eq. 5). From eq. 1 and with the input voltages $Vx = Vy = +10V$ the transfer function of the LH0094 becomes

$$E_o = 10 \times (Vz/10)^m. \qquad \text{(eq. 6)}$$

From eq. 4 it is known that $m = 2.827$

The value of m is set by R1 and R2, with

$$m = (R1 + R2)/R1$$

(from the datasheet) and the recommendation that

$$R1 + R2 = 200\Omega$$

(approximately). Suitable values are

$$R1 = 127.89\Omega,$$

$$R2 = 70\Omega.$$

R3 and the potentiometer R4 in *Figure 2* serve to trim the maximum output E_o to +10V when the input voltage Vz = +10V.

Both Vx and Vy should be connected to a regulated +10V, since the accuracy of this voltage directly influences the accuracy of the output voltage.

3b. Circuit to Realize the Sine Function

Figure 3 shows the schematic for the complete circuit. Q1 is the LH0094 connected as shown in *Figure 2* to perform the exponential term of the transfer function (eq. 5). The output voltage E_o is amplified by Q2, which also adds some fraction of the input voltage Vz coming through C and D, thus accommodating the linear term of the desired transfer function (eq. 5).

4. TRANSFER FUNCTIONS

The transfer function of the circuit in *Figure 3* is

$$V_{OUT} = -E_o \frac{B}{A} + Vz \times \frac{D}{C+D} \times \frac{A+B}{A}$$

$-B/A$ is the inverting voltage gain, $(A+B)/A$ is the non-inverting voltage gain. $Vz \times D/(C+D)$ is the fraction of the input voltage which is fed into the non-inverting input.

With eq. 6 used to substitute for E_o, the transfer function becomes

$$V_{OUT} = -10 \frac{Vz^m}{10^m} \times \frac{B}{A} + Vz \frac{D}{C+D} \times \frac{A+B}{A} \qquad \text{(eq. 7)}$$

The transfer function to be realized is (from eq. 5):

$$V_{OUT} = Vz \times \pi/2 - Vz^m \times 10 \times (\pi/20)^m/6.28 \qquad \text{(eq. 8)}$$

5. DIMENSIONING THE CIRCUIT PARAMETERS

For the two transfer functions, eq. 7 and eq. 8, to be identical, in both functions the coefficients of Vz^m must be equal, and also the coefficients of Vz must be equal to each other.

$$-10 \frac{Vz^m}{10^m} \times \frac{B}{A} = -10 \times (\pi/20)^m/6.28$$

With $m = 2.827$ (eq. 4) this results in

$$B/A = 0.5708. \qquad \text{(eq. 9)}$$

The linear coefficients set equal give

$$\frac{D}{C+D} \times \frac{A+B}{A} = \pi/2$$

With the known value for B/A (eq. 9) this yields

$$C/D = 0 \qquad \text{(eq. 10)}$$

or D = infinite, or open, C having a finite value.

6. DIMENSIONING OF THE RESISTORS

10 kΩ is a good first choice for both load and source impedances. Load impedances need to be larger than 2 kΩ. Sources driving Q2 can be quite high, if the LF411 is chosen, because of its FET input. However, the impedances should not be too high, because of the time constants formed together with the circuit capacitances. Below 1 MΩ this concern will not arise.

If A is chosen as A = 10 kΩ,
B becomes (eq. 9) ... B = 0.5708 A = 5.708 kΩ.
D is infinite,
C is chosen so it is approximately equal to
$A \times B/(A + B) = 3.6$ kΩ.

In this case both inputs of the op-amp Q2 see the same impedance.

7. LIST OF RESISTORS

R1 = 127.89	A = 10k
R2 = 70	B = 5.708k
R3 = 2 M	C = 3.6k, 5%
R4 = 10k POT	D = open

II. REALIZING THE FUNCTION y = cos x

1. OBJECTIVE

A circuit needs to be designed where the output voltage is the cosine of the input voltage, as shown in *Figure 1*. The basic function is

$$y = \cos x.$$

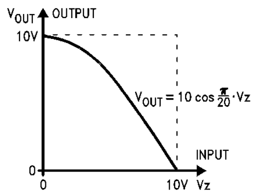
If substitutions are made: $x = c \times Vz$ and $y = V_{OUT}/k$, the function becomes

$$V_{OUT} = k \times \cos c \times Vz \qquad \text{(eq. 11)}$$

This equation connects input and output voltage by a cosine function (see *Figure 4*). The values of k and c can be determined so that these boundary conditions are met:

at $V_{IN} = Vz = 0V \ldots V_{OUT} = 10V$ and the argument $c \times Vz = 0$ radians, at $V_{IN} = Vz = 10V \ldots V_{OUT} = 0V$ and the argument $c \times Vz = \pi/2$ rad.

Both k and 1/c have the dimension of a voltage, and therefore the value of the cosine (V_{OUT}/k) and its argument ($c \times Vz$) remain without dimension. In the following equations all voltages need to be measured in Volts.



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The independent variable has been named Vz to conform with the convention of the LH0094. Both Vy and Vz are voltages.

FIGURE 4. Cosine Function to be Realized

2. DEVELOPMENT OF A REALIZABLE FUNCTION

From eq. 2 and its boundary conditions it can be shown that

$$c = \frac{1}{10V} \times \frac{\pi}{2} \quad \text{and} \quad k = 10V$$

The desired function (eq. 2) becomes

$\frac{V_{OUT}}{10V} = \cos \frac{\pi}{2} \times \frac{Vz}{10V}$, or, in a shorter form,

$$V_{OUT} = 10 \cos (Vz \times \pi/20), \qquad \text{(eq. 12)}$$

where Vz is the input voltage.

Because the LH0094 can only create an exponential function, the cosine function in eq. 12 needs to be substituted by a polynomial.

The approximation

$$\cos x = 1 + 0.2325\, x - x^m/1.445, \qquad \text{(eq. 13)}$$

with $x = Vz \times \pi/20$ and $m = 1.504$,

allows the desired function to be written as

$$V_{OUT} = 10\, [\, 1 + 0.2325\, Vz \times \pi/20 - (Vz \times \pi/20)^m/1.445\,]. \qquad \text{(eq. 14)}$$

This approximation is accurate for Vz from 0V to +10V with a theoretical accuracy of 0.75%.

3. DEVELOPMENT OF THE CIRCUIT

3a. Realization of the Exponential Term

The LH0094 is used to realize a function proportional to $(Vz)^m$, which will then be scaled and added to the other voltages which correspond to the remainder of the terms in eq. 13.

From eq. 1, with Vx and Vy held constant at +10V, the transfer function of the LH0094 becomes

$$E_o = 10\, (Vz/10)^m. \qquad \text{(eq. 15)}$$

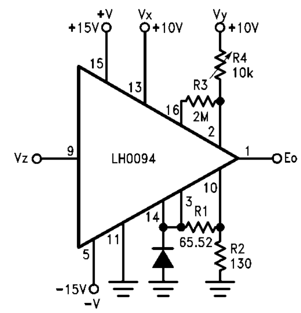
Figure 5 shows how to connect the LH0094 to perform the exponential portion of the desired transfer function (eq. 14). The resistors R1 and R2 set the value of m (from the datasheet):

$$m = (R1 + R2)/R1.$$

For $m = 1.504$ (eq. 13), and the recommendation that $R1 + R2$ be approximately 200Ω, suitable values are R1 = 65.52Ω and R2 = 130Ω.

R3 and the potentiometer R4 in *Figure 2* serve to trim the maximum output E_o to 10V when the input voltage Vz = 10V.

Both Vx and Vy should be connected to a regulated +10V, since the accuracy of this voltage directly influences the accuracy of the circuit.



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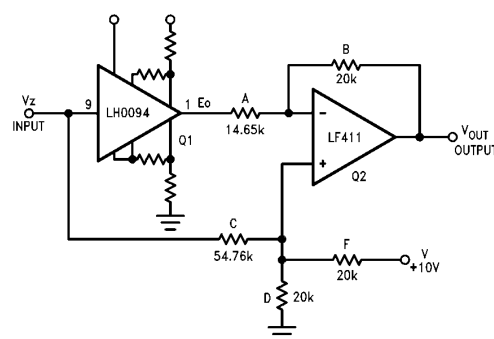
The clamp diode is for protection. R4 trims the output voltage to 10V when all three inputs Vx, Vy, and Vz, are at +10V.

FIGURE 5. The LH0094 Connected to Output the Function $E_o = 10 \times (Vz/10)^{1.504}$

3b. Circuit to Realize the Cosine Function

Figure 6 shows the schematic for the complete circuit. Q1 is the LH0094 connected as shown in *Figure 5* to perform the exponential term of the desired transfer function (eq. 14). The output voltage E_o is amplified by Q2, which also adds some fraction of the input voltage Vz coming through C, as well as a constant voltage coming through R.



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Q1 is the LH0094 connected as per *Figure 5*. Its output is proportional to the exponential term in the transfer function.

FIGURE 6. Circuit to Realize the Cosine Function

4. TRANSFER FUNCTIONS

The transfer function of this circuit *(Figure 3)* is

$$V_{OUT} = -E_o \frac{B}{A} + \frac{A+B}{A}\left(Vz \frac{R \parallel D}{C+R \parallel D} + V \frac{C \parallel D}{F+C \parallel D}\right),$$

where $R \parallel D = R \times D/(R+D)$, and $C \parallel D = C \times D/(C+D)$. The factor $(-B/A)$ is the inverting voltage gain, $(A+B)/A$ the non-inverting voltage gain. The term $z \times (R \parallel D)/(C+R \parallel D)$ describes the fraction of the input voltage fed into the non-inverting input, and $V \times (C \parallel D)/(F+C \parallel D)$ is the DC voltage at the non-inverting input. The op-amp sums and amplifies all three voltages.

With eq. 15 used to substitute for E_o, the transfer function of the circuit becomes

$$V_{OUT} = -10 \frac{Vz^m}{10^m} \times \frac{B}{A} + \frac{A+B}{A} \left(Vz \frac{R \parallel D}{C + R \parallel D} + V \frac{C \parallel D}{F + C \parallel D}\right) \quad \text{(eq. 16)}$$

The transfer function to be realized is (eq. 14):

$V_{OUT} = 10\,[1 + 0.2325\, Vz \times \pi/20 - (Vz \times \pi/20)^m/1.445]$.

5. DIMENSIONING OF THE CIRCUIT PARAMETERS

For the two transfer functions (eq. 14, 16) to be identical, in both functions the coefficients of Vz^m must be equal, the coefficients of Vz must be equal, and the constants must be equal.

The coefficients of Vz^m set equal give

$$-10 \frac{Vz^m}{10^m} \times \frac{B}{A} = -10\,(\pi/20)^m/1.445$$

This yields $B/A = 1.365$. (eq. 17)

The linear coefficients set equal give

$$\frac{A+B}{A} \times \frac{F \parallel D}{C+F \parallel D} = 10\,(0.2325) \times \frac{\pi}{20} \quad \text{(eq. 18)}$$

This yields

$$\frac{C}{F \parallel D} = \left(\frac{B}{A} + 1\right) \times \frac{\pi}{2} \times \frac{1}{0.2325} = 5.475 = a \quad \text{(eq. 19)}$$

The coefficients of the constants set equal give

$$\left(\frac{B}{A} + 1\right) \times V \times \frac{C \parallel D}{F+C \parallel D} = 10, \text{ or, with } V = 10V,$$

$$\frac{F}{C \parallel D} = \left(\frac{B}{A} + 1\right) \times \frac{10}{10} - 1 = 1.3649 = b \quad \text{(eq. 20)}$$

Eq. 19 and eq. 20 are two equations for the three unknowns C, D, and R. This means one of the unknowns can be chosen. This makes sense, because in voltage dividers it is the ratio of resistors that counts. The terms a and b are used as abbreviations for easier use.

The resistor D will be chosen and considered known. With this premise it can be shown the equations 19 and 20 yield

$C = D \times (a \times b - 1)/(b + 1) = 2.7373\,D$ (eq. 21)

$F = D \times (a \times b - 1)/(a + 1) = 0.9997\,D$ (eq. 22)

6. DIMENSIONING OF THE RESISTORS

10 kΩ is a good choice for both load and source impedances. Load impedances need to be larger than 2 kΩ. Sources driving Q2 can be quite high, if the LF411 is chosen, because of its FET input. However, the impedances should not be too high, because of the time constants formed together with the circuit capacitances. Below 1 MΩ this concern will not arise.

6a. First Round of Dimensioning

If A is chosen, as first try, $A1 = 10$ kΩ,
B becomes (eq. 17) $B1 = 1.3649\, A1 = 13.649$ kΩ.
D is chosen as $D1 = 10$ kΩ,
C then becomes (eq. 21) $C1 = 2.7373\, D1 = 27.373$ kΩ,
and F becomes (eq. 22) $F1 = 0.99968\, D1 = 9.9968$ kΩ

6b. Second Round of Dimensioning

The values of the resistors are in the proper range as far as loading and input impedance are concerned. However, for reasons of accuracy the amplifier Q2 needs source impedances on its inverting and non-inverting inputs that are approximately equal.

$A1 \parallel B1 = A1 \times B1/(A1+B1) = 5.772$ kΩ
$C1 \parallel D1 \parallel F1 = 1/(1/C1 + 1/D1 + 1/F1) = 4.227$ kΩ

These values are relatively close to each other, but it is easy to make them even closer. The scaling factor is $5.772/4.227 = 1.3653$, and with it the new values for C, D and F become

$C2 = 1.365\, C1 = (1.365) \times (27.373) = 37.373$ kΩ,
$D2 = 1.365\, D1 = (1.365) \times (10) = 13.653$ kΩ
$F2 = 1.365\, F1 = (1.365) \times (9.9968) = 13.649$ kΩ

The resistors A1 and B1 remain unchanged:

$A2 = A1 = 10$ kΩ
$B2 = B1 = 13.649$ kΩ

6c. Final Dimensions

Three resistors have about the same value: B1, D2, and F2. If they are all made 13.65 kΩ the error is 0.02%. This error will in most cases be negligible. To get standard values the three resistors can be set 20 kΩ. The remaining resistors A1 and C2 will be proportionally scaled with a factor of $20/13.65 = 1.4652$. Since all resistors are scaled by the same factor, all the previously established relationships, like voltage attenuation and impedance matching, remain unchanged.

7. LIST OF RESISTORS

R1 = 65.52	A = 14.652k
R2 = 130	B = 20k
R3 = 2 M	C = 54.759k
R4 = 10k POT	D = 20k
	F = 20k

For good accuracy the voltage $V = +10V$ needs to be regulated and is best connected to Vx and Vy, which need to be regulated as well.

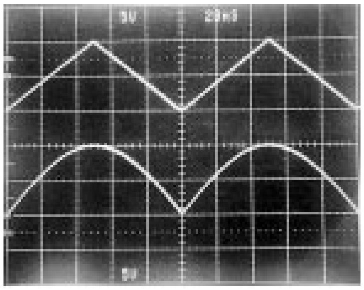

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FIGURE 7. Input and Output Voltage of the Sine Circuit

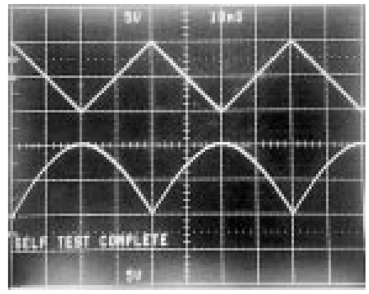

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FIGURE 9. Input and Output Voltage of the Cosine Circuit

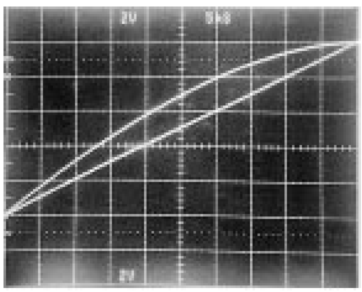

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FIGURE 8. Input and Output Voltage of the Sine Circuit, Detail

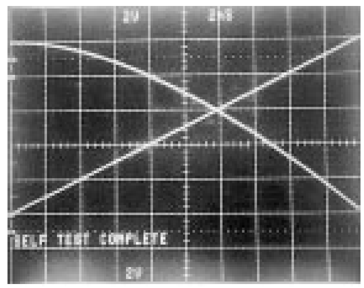

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FIGURE 10. Input and Output Voltage of the Cosine Circuit, Detail

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