APPROXIMATIONS

FOR THE CONTROL DATA 1604

by Hans J. MAEHLY

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HANS J. MAEHLY*

* Presently Princeton University mailing address: Institute for Advanced Study--E. C. P., Princeton, New Jersey. After 1 February 1960: Mathematics Department, Syracuse University, Syracuse 10, New York.

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PREFACE

The CONTROL DATA CORPORATION extends its sincerest appreciation to Dr. Hans J. Maehly for the task he has performed in preparing this report. The contents represent, we believe, a significant step forward in the development of numerical techniques.

Presented here is a (slightly) edited reproduction of Dr. Maehly's original text. An attempt was made to retain as much of the flavor of the initial report as possible.

K. H. Olson
Supervisor of Applications and Analysis
March, 1960

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CONTENTS

INTRODUCTION AND MACHINE CHARACTERISTICS

The purpose of this report is to describe and justify several approximations for the elementary non-rational functions which are, in our opinion, particularly suited for the Control Data 1604 computer. The manual for the computer has been carefully consulted since some of the machine characteristics have a decisive influence on the selection of best approximations. The most important of these characteristics are:

(i) Control Data 1604 is a BINARY COMPUTER

(ii) 1 word = 48 bits: sign + 47 bits for fixed point

= exponent (11) + (sign + 36) bits for floating point

Thus basic round-off $\leq \epsilon_o = 2^{-48} = 3.55 \times 10^{-15}$ (fixed)

$\leq \epsilon_1 = 2^{-37} = 7.28 \times 10^{-12}$ (floating)

(iii) Average execution times are about: (in microseconds)

Fixed point	Add 7	multiply 45	divide 65
Floating Point	Add 19	multiply 45	divide 56

(iv) Size of memory: 32768 words of core, all equally accessible. (Most customers will also use tape units)

Conclusions:

These machine characteristics will have the following effects on subroutines for special functions:

(i) The basic ranges for such functions as exp (X), log X, $\sqrt{X}$ and $X^{1/3}$ are quite small; further reduction will not be necessary.

(ii) For fixed point subroutines, the truncation error λ_o should be smaller or about equal to ϵ_o:

$$\lambda_o \lesssim 3.55 \cdot 10^{-15}$$

though $\lambda_o \approx 10^{-14}$ may be acceptable.

For floating point subroutines, the relative error λ_1 should be at most ϵ_1:

$$\lambda_1 \leq 7.28 \cdot 10^{-12}$$

Internal round-off can be reduced by coding the subroutine internally in fixed point, using (some of) the 11 exponent bits.

(iii) Division time $\approx$ 3/2 multiplication time; therefore, fractional approximations can be used to great advantage.

(iv) Though it is always desirable to make subroutines short, this restriction is not quite so serious with a 32,768 word memory as with 2000 or 4000 words. Even a short table of key values may be considered if this helps to save time. Extensive tables, however, should in general be avoided.

1. APPROXIMATIONS FOR THE SQUARE ROOT

(1.1) Range:

For a floating point subroutine, the exponent of X will be separated from the mantissa and the two cases, "exponent even" and "exponent odd", will be treated separately. The latter case is equivalent to "$\frac{1}{4} \leq$ mantissa $\leq \frac{1}{2}$".

For a fixed point subroutine, the number will be "half-normalized" by an even number 2n of left-shifts and the two cases are then $\frac{1}{2} \leq X \cdot 2^{2n} < 1$ and $\frac{1}{4} \leq X \cdot 2^{2n} < \frac{1}{2}$.

These two ranges can be treated jointly or separately. Separation means better initial approximation and may save one iteration, depending on the type of initial approximation used and on the accuracy required.

(1.2) The Newton Iteration Formula

The theory of rational approximations to the square root can be understood best on the basis of the Newton Iteration Formula:

If Y_i is an approximation to $Y = \sqrt{X}$,

then

$$Y_{i+1} := \frac{1}{2} \left(Y_i + \frac{X}{Y_i}\right) \text{ *)} \tag{1.2.1}$$

will be a better approximation. Let δ_i be the "relative"

*) Throughout this report, the notation (:=) stands for "is defined by" (cf. ALGOL).

or logarithmic error of Y_i, viz.

$$\delta_i := \ln (Y_i/Y) \qquad (1.2.2)$$

The logarithmic error of Y_{i+1} will then be

$$\delta_{i+1} := \ln (Y_{i+1}) = \ln (\cosh \delta_i) \qquad (1.2.3)$$

whence

$$\delta_{i+1} \approx \tfrac{1}{2} \delta_i^2 \qquad \text{if } \delta_i \ll 1 \ . \qquad (1.2.4)$$

This is the Newton Iteration in its standard form. It can be improved as follows: If the maximum of $|\delta_i|$ is known (for a given interval $[X_1, X_2]$),

$$\lambda_i = \max_{[X_1,\ X_2]} |\delta_i| \qquad (1.2.5)$$

Then the maximum of $|\delta_{i+1}|$ can be halved by redefining $\overline{Y}_{i+1} := \dfrac{Y_i + (X/Y_i)}{2 \sqrt{\cosh \lambda_i}}$ (1.2.6)

so that

$$\overline{\lambda}_{i+1} := \max_{[X_1,\ X_2]} |\overline{\delta}_{i+1}| = \tfrac{1}{2} \ln (\cosh \lambda_i) \qquad (1.2.7)$$

It will be noted, however, that this improvement by merely a factor 2 requires a true multiplication, while the original iteration formula does not, since division by 2 can be done by a right shift. Therefore, <u>we shall use the original Newton formula for iteration</u>. The improved formula, however, immediately leads to the best linear approximation.

(1.3) Best Linear Approximation

In order to find the best linear approximation, we start out with the best constant Y_o and apply one improved Newton Iteration. Obviously,

$$Y_o := (X_1 \; X_2)^{\frac{1}{4}} \tag{1.3.1}$$

is the "best" constant approximation to $\sqrt{X}$ for the interval $[X_1, X_2]$, yielding the maximum error

$$\lambda_o = \max_{[X_1,\, X_2]} \left| \ln (Y_o/Y) \right| = \tfrac{1}{4} \ln \frac{X_2}{X_1} \tag{1.3.2}$$

One "improved Newton step" yields

$$\bar{Y}_1 = \frac{Y_o + X/Y_o}{2\sqrt{\cosh \lambda_o}} = a + bX$$

with

$$a := \frac{(X_1 \; X_2)^{\frac{1}{4}}}{\sqrt{2\left[(X_2/X_1)^{\frac{1}{4}} + (X_1/X_2)^{\frac{1}{4}}\right]}}$$

$$b := \frac{(X_1 \; X_2)^{-\frac{1}{4}}}{\sqrt{2\left[(X_2/X_1)^{\frac{1}{4}} + (X_1/X_2)^{\frac{1}{4}}\right]}} \tag{1.3.3}$$

The relative error of this approximation is:

$$\bar{\lambda}_1 = \max_{[X_1,\, X_2]} \left| \ln (\bar{Y}_1/Y) \right| = \tfrac{1}{2} \ln \left[\frac{(X_2/X_1)^{\frac{1}{4}} + (X_1/X_2)^{\frac{1}{4}}}{2} \right] \tag{1.3.4}$$

or approximately

$$\bar{\lambda}_1 \approx \left[\frac{\ln (X_2/X_1)}{8} \right]^2 \tag{1.3.5}$$

It may be desirable (e.g. for scaling reasons) to have no error at the ends of the interval, i.e., for $X = X_1$ and $X = X_2$. The solution is:

$$\hat{Y}_1 = \hat{a} + \hat{b}X \quad \left[\begin{array}{l} \hat{a} := \dfrac{\sqrt{X_1 \cdot X_2}}{\sqrt{X_1} + \sqrt{X_2}} \\ \\ \hat{b} := \dfrac{1}{\sqrt{X_1} + \sqrt{X_2}} \end{array} \right] \tag{1.36}$$

and the maximum error is exactly doubled:

$$\hat{\lambda}_1 = \ln\left[\frac{\left(\frac{X_2}{X_1}\right)^{\frac{1}{4}} + \left(\frac{X_1}{X_2}\right)^{\frac{1}{4}}}{2}\right] \approx 2\left[\frac{\ln (X_2/X_1)}{8}\right]^2 \tag{1.3.7}$$

A few numerical values are given below, including the errors after 1, 2 and 3 iterations (Standard Newton Iterations):

	$X_2/X_1 = 4$		$X_2/X_1 = 2$	
	$\hat{\lambda}_i$	$\bar{\lambda}_i$	$\hat{\lambda}_i$	$\bar{\lambda}_i$
Initial error λ_1	$5.89\ 10^{-2}$	$2.95\ 10^{-2}$	$1.49\ 10^{-2}$	$7.47\ 10^{-3}$
1 Iteration λ_2	$1.74\ 10^{-3}$	$4.34\ 10^{-4}$	$1.11\ 10^{-4}$	$2.79\ 10^{-5}$
2 Iterations λ_3	$1.51\ 10^{-6}$	$9.40\ 10^{-8}$	$6.16\ 10^{-9}$	$3.89\ 10^{-10}$
3 Iterations λ_4	$1.14\ 10^{-12}$	$4.42\ 10^{-15}$	$1.89\ 10^{-17}$	$7.57\ 10^{-20}$

(1.3.8)

Conclusions: Three iterations are needed after a linear initial approximation but the range $|\frac{1}{4}, 1|$ need not be separated into two smaller ones. $\hat{\lambda}_4$ is good enough for floating point, $\bar{\lambda}_4$ is just fine for fixed point.

(1.4) A Simple Fractional Approximation

The linear approximation $Y_1 = a + bX$ requires 1 addition + 1 multiplication, a fractional approximation of the form

$$Y_1 := a + \frac{b}{c + X} \qquad (1.4.1)$$

requires little more, viz., 2 additions and 1 division. This investment will pay off if we can thereby save one iteration. This is indeed the case.

It can easily be shown from the general theory of approximation that we can expect our new Y_1 (X) to equal $\sqrt{X}$ at <u>three</u> points rather than two (for the linear case). For reasons of scaling, viz., to avoid spill when computing the square root of $1 - 2^{-47}$, we shall choose 1 as one of these points. It can also be shown that the third value of X, say X_3, where $Y_1(X) = \sqrt{X}$, must be the square of the second, if the relative error is to be minimized. Therefore, we choose:

$$a + \frac{b}{c + X} = \sqrt{X} \quad \text{for} \quad \begin{bmatrix} X_1 = 1 \\ X_2 = \alpha^2 \\ X_3 = \alpha^4 \end{bmatrix} \qquad (1.4.2)$$

The solution is relatively simple:

$$\left. \begin{aligned} a &= 1 + \alpha + \alpha^2 \\ b &= -[\alpha + 2\alpha^2 + 2\alpha^3 + 2\alpha^4 + \alpha^5] \\ &= -(1 + \alpha)^2 (1 + \alpha^2) \cdot \alpha \\ c &= \alpha + \alpha^2 + \alpha^3 = \alpha \cdot a \end{aligned} \right] \qquad (1.4.3)$$

For a fixed point square root routine, I recommend the following initial approximations:

(i) UPPER RANGE: $\frac{1}{2} \leqslant X < 1$

Fitting points: $X_1 = 1$, $X_2 = \alpha^2$, $X_3 = \alpha^4$

$\alpha = \frac{109}{128}$ In order to avoid binary round-off of the constants

error: $\lambda_{rel} = 4.0 \times 10^{-4}$

$$a_1 = +\frac{42217}{16384} \qquad \frac{a_1}{4} = +.511644 \text{ (octal)}$$

$$b_1 = -\frac{17\ 30502\ 29565}{3\ 43597\ 38368} \qquad \frac{b_1}{16} = -.2411246171475 \text{ (octal)} \qquad (1.4.4)$$

$$c_1 = +\frac{46\ 01653}{20\ 97152} \qquad \frac{c_1}{4} = +.43067152 \text{ (octal)}$$

(ii) LOWER RANGE: $\frac{1}{4} \leqslant X < \frac{1}{2}$

Fitting Points: $X_1 = \frac{1}{4}$, $X_2 = \frac{\alpha^2}{4}$, $X_3 = \frac{\alpha^4}{4}$

$\alpha = \frac{151}{128}$

error: $\lambda_{rel.} = 4.3 \times 10^{-4}$

After an easy transformation we obtain:

$$a_2 = +\frac{58513}{32768} \qquad \frac{a_2}{2} = +.711104 \text{ (octal)}$$

$$b_2 = -\frac{46\ 05801\ 37335}{27\ 48779\ 06944} \qquad \frac{b_2}{4} = -.32636267\ 122734 \text{ (octal)} \qquad (1.4.5)$$

$$c_2 = +\frac{59\ 16935}{83\ 88608} \qquad \frac{c_2}{2} = +.26444407 \text{ (octal)}$$

The octal forms have already been scaled for the following recommended algorithms:

UPPER RANGE || LOWER RANGE

$$\frac{(Y_1)}{2} := \left[\frac{a_1}{4} + \frac{(b_1/16)}{\frac{(c_1)}{4} + X/4}\right] 2 \quad \Bigg\| \quad \frac{Y_1}{2} := \frac{(a_2)}{2} + \frac{(b_2/4)}{(c_2/2) + X/2}$$

$$Y_2 := \frac{X/4}{Y_1/2} + Y_1/2 \qquad (1.4.5)$$

$$Y_3 := \frac{X/2}{Y_2} + Y_2/2$$

All division by 2 and 4 should be executed as unrounded right-shifts. No spill should occur, I think, even if X is very close to 1, such as $X = 1 - 2^{-47}$ or $X = 1 - 2^{-46}$

Accuracy: The relative errors for Y_1, Y_2, Y_3 are:

UPPER RANGE	LOWER RANGE
$\lambda_1 = 4.0 \cdot 10^{-4}$	$\lambda_1 = 4.3 \cdot 10^{-4}$
$\lambda_2 = 8.0 \cdot 10^{-8}$	$\lambda_2 = 9.4 \cdot 10^{-8}$
$\lambda_3 = 3.2 \cdot 10^{-15}$	$\lambda_3 = 4.4 \cdot 10^{-15}$

maximum absolute error $\lambda_{abs.} = 3.2 \cdot 10^{-15}$

Compare: $2^{-48} = 3.5 \cdot 10^{-15}$ (1.4.6)

Note: I assume that these data are correct and reasonably accurate, but I did not have a chance to check them as carefully as I should like to.

(1.5) Fractional Approximation for Floating Point

Since somewhat less accuracy is required for floating point, a simple fractional initial approximation of the form (1.4.1) may be used for the "full range", as e.g. $\left[\frac{1}{4}, 1\right]$. It is more convenient for deriving the formulae below, to treat the range

$$\tfrac{1}{2} \leq X \leq 2 \qquad (1.5.1)$$

Since there are no scaling difficulties in floating point, and since the point X = 1, being in the logarithmic center of the interval will be one in which our initial approximation will be exact, we can use an exact best-fit approximation for this interval, minimizing the relative error, i.e. minimizing

$$\lambda_1 := \max_{\left[\frac{1}{2},\, 2\right]} \left| \ln\left(Y_1 / \sqrt{X}\right)\right| \qquad (1.5.2)$$

where

$$Y_1 := a + \frac{b}{c + X} \qquad (1.5.3)$$

It can be shown that the constants a, b, c can be computed as follows: *)

$$\left.\begin{aligned} v &:= (X_{max} + 1)/(X_{max} - 1) = 3 \\ w &:= \left[4v^2\,(v^2-1)\right]^{1/3} \cong 6.60385\ 4497 \\ z &:= \sqrt{w^2 - w + 1} = \sqrt{\frac{w^3+1}{w+1}} \cong 6.16498\ 4974 \\ m &:= \tfrac{1}{2} + \frac{3/4}{\left[\sqrt{w+1} + \sqrt{2z - w + 2}\right]\left[z + w - \tfrac{1}{2}\right]} = .51104\ 01655 \\ a &= c = \frac{1+m}{1-m} \cong 3.09031\ 5520 \\ b &= 1 - a^2 \cong -8.55005\ 0013 \end{aligned}\right] \qquad (1.5.4)$$

*) Derivation published in Math. Comp. 1960

Note: While the exact numerical value of m is not critical (it should rather be chosen a trifle too big, but not smaller than the exact value), the relation $b = 1 - a^2$ should be exactly fulfilled. I recommend, therefore, to truncate m to, say, 17 binaries (rounding up) and then to compute $b = 1 - a^2$ (by machine).

Error bounds: The maximum relative error of the approximation (1.5.3) - (1.5.4) is

$$\lambda_1 = 2.52614\cdot 10^{-3}$$

whence

$$\left.\begin{aligned} \lambda_2 &\cong 3.19\cdot 10^{-6} \\ \lambda_3 &\cong 5.09\cdot 10^{-12} \end{aligned}\right] \qquad (1.5.5)$$

Compare: $2^{-37} = 7.28\cdot 10^{-12}$

To shift the range from $\left[\frac{1}{2},\ 2\right]$ to $\left[\varkappa/2,\ 2\varkappa\right]$, where $\varkappa$ is an arbitrary number, take

$$\left.\begin{aligned} a &= \frac{1+m}{1-m}\cdot\varkappa^{\frac{1}{2}} \\ c &= \frac{1+m}{1-m}\cdot\varkappa \\ b &= \left[1-\left(\frac{1+m}{1-m}\right)^2\right]\varkappa^{3/2} \end{aligned}\right\} \quad \text{for } \frac{\varkappa}{2} \leq x \leq 2\varkappa \qquad (1.5.6)$$

In particular, if

$$a_1 := \frac{1+m}{1-m} \text{ and } b_1 := -a_1^2 + 1 \qquad (1.5.7)$$

(with m = .51104 01655)

then

$$\left.\begin{aligned} a &= a_1\cdot 2^{n} \\ c &= a_1\cdot 2^{2n} \\ b &= b_1\cdot 2^{3n} \end{aligned}\right] \text{ for } 2^{2n-1} \leq x \leq 2^{2n+1} \qquad (1.5.7)$$

2. APPROXIMATIONS FOR EXP (X)

All approximations in this section are based on the well-known continued fraction

$$e^X = 1 + \frac{2X|}{|2-X} + \frac{X^2|}{|6} + \frac{X^2|}{|10} + \frac{X^2|}{|14} + \frac{X^2|}{|18} + \frac{X^2|}{|22} + \cdots \qquad (2.1)$$

which can also be written in the form

$$e^X = 1 + \frac{2X}{S(X^2) - X} = \frac{S(X^2) + X}{S(X^2) - X} \qquad (2.2)$$

where

$$S(X^2) = X \operatorname{cth} \frac{X}{2} = 2 + \frac{X^2|}{|6} + \frac{X^2|}{|10} + \frac{X^2|}{|14} + \cdots \qquad (2.3)$$

The first four approximants to $S(X^2)$ are:

$$S_0 := 2 \qquad (2.4.0)$$

$$S_1 := 2 + \frac{X^2}{6} \qquad (2.4.1)$$

$$S_2 := 2 + \frac{X^2}{6 + \frac{X^2}{10}} \equiv 12 - \frac{600}{60 + X^2} \qquad (2.4.2)$$

$$S_3 := 2 + \frac{X^2}{6 + \frac{X^2}{10 + \frac{X^2}{14}}} \equiv 2 + X^2 \left(.05 + \frac{4.9}{42 + X^2}\right) \qquad (2.4.3)$$

The last expressions in (2.4.2) and (2.4.3) are the forms which can be evaluated most quickly.

Range: It is well known that for a binary machine in floating point operation, the range of X can easily be reduced to

$$|X| \leq \tfrac{1}{2} \ln 2 \qquad (2.5)$$

(cf. e.g. (2.10) below)

If we approximate e^X by

$$R_3(X) := \frac{S_3(X^2) + X}{S_3(X^2) - X} \tag{2.6}$$

then the maximum relative error will be

$$\lambda_3 := \left| \ln \left[e^{-X} \cdot R_3(X) \right] \right| \cong 2.8 \times 10^{-12} \quad \text{for } |X| = \tfrac{1}{2} \ln 2 \tag{2.7}$$

While this is just good enough for a floating point exponential subroutine it may be worth while to note that for the same range the best-fit approximation R_3^*, viz.,

$$R_3^*(X) := \frac{S_3^*(X^2) + X}{S_3^*(X^2) - X} = 1 + \frac{2X}{S_3^*(X^2) - X}$$

with

$$S_3^*(X^2) := a + X^2 \left(b + \frac{c}{d + X^2} \right) \tag{2.8}$$

and

$a = 2.00000\ 00000\ 00575\ 924$

$b = .04996\ 24891\ 36450\ 764$

$c = 4.90315\ 47989\ 68682\ 648$

$d = 42.01353\ 28950\ 41661\ 680$

reduces the error by a factor close to 256, thus:

$$\lambda_3^* := \max \left| \ln \left[e^{-X} R_3^*(X) \right] \right| \cong 1.11 \times 10^{-14} \quad \text{for } |X| \leq \tfrac{1}{2} \ln 2 \tag{2.9}$$

Note: The above-mentioned reduction of the range to $|X| \leq \frac{1}{2} \ln 2$ is achieved as follows: To compute e^u, find the integer n so that

$$u = n \ln 2 + X, \quad |X| \leq \frac{1}{2} \ln 2$$

thus (2.10)

$$e^u = 2^n e^X$$

This n is simply added to the exponent of the result.

The only practical way to find n (for a general purpose subroutine) is to multiply u by (1/ln 2) and then to determine the nearest integer. If [ ] denotes "integer part of" this can be written as follows:

$$z := u \cdot \left(\frac{1}{\ln 2}\right) \tag{2.11}$$

$$n := \left[z + \tfrac{1}{2}\right] \tag{2.12}$$

$$w := z - n \tag{2.13}$$

$$X := w \cdot \ln 2 \tag{2.14}$$

Comments: The multiplication (2.11) is due to the fact that we want e^X, while the machine has base 2. For many applications base 2 is just as good; for example, if the logarithmic and exponential subroutines are used to compute odd (or high) powers such as $X^{7/3}$ or X^{15}, i.e. in all those cases where logarithm and "antilogarithm" are used as auxiliary functions, just like $\log_{10}$ and 10^X are often used for numerical computations without an automatic computer.

I therefore, recommend that the basic subroutine computes 2^X and that the division by ln 2 (multiplication by 1/ln 2) is executed outside, if necessary, or is done automatically as an option (Separate entry to basically the same subroutine).

The multiplication (2.14) can also be avoided since

$$2^w = e^X = e^{w\ln 2} \approx \bar{R}(w)$$

if

$$R(w) := \frac{\bar{S}(w^2) + w}{\bar{S}(w^2) - w} = \frac{2w}{\bar{S}(w^2) - w} + 1 \qquad (2.15)$$

with

$$\bar{S}(w^2) := \bar{a} + w^2 \left(\bar{b} + \frac{\bar{c}}{\bar{d} + w^2}\right)$$

$$\begin{aligned} \bar{a} &:= a/\ln 2 \\ \bar{b} &:= b \cdot \ln 2 \\ \bar{c} &:= c/\ln 2 \\ \bar{d} &:= d/(\ln 2)^2 \end{aligned} \qquad (2.16)$$

The numerical values of a, b, c, d are the same as in (2.8), but those of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ have not yet been computed (on a normal desk computer, double precision is mandatory; on the CDC 1604 full fixed point precision will just be sufficient, at least for $\bar{b}$, $\bar{c}$, $\bar{d}$.) This reduces the number of multiplications (M) and divisions (D) to 2M + 2D for 2^z and 3M + 2D for e^u. The entire subroutine will take around 400 μsec.

3. APPROXIMATIONS FOR THE LOGARITHMIC FUNCTION

Note: Just as a subroutine for 2^X is somewhat shorter and simpler than one for e^X, the same will be true for $\log_2 X$ as compared to the natural logarithm ln X. Therefore, we shall assume here that the subroutine proper will compute $\log_2 X$. The final multiplication

$$\ln X = (\ln 2) \log_2 X \tag{3}$$

can be done outside the subroutine or it will be "an optional extra at additional cost".

(3.1) Reduction of the Range

If X is represented in the machine as a normalized floating-point number, then the integer part of the logarithm will be the exponent of X minus one; or if

$$X = \xi \; 2^n, \quad \tfrac{1}{2} \leq \xi < 1 \tag{3.1.1}$$

then

$$Y := \log_2 X = n + m, \; m := \log_2 \xi \tag{3.1.2}$$

This reduces the range for which $\log_2 \xi$ must be computed to the interval $\left[\tfrac{1}{2}, 1\right]$.

It is well-known that, for any given range of the argument of the logarithmic function, the series

$$\ln \xi = \ln \frac{1 + t}{1 - t} = 2t \left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \frac{t^6}{7} + \cdots\right) \tag{3.1.3}$$

converges much better than

$$\ln \xi = \ln (1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots \tag{3.1.4}$$

The same is true for the corresponding continued fractions and for best-fit approximations of polynomial or fractional form. The maximum absolute value of t can further be reduced for the interval

$$\xi_{min.} \leqslant \xi \leqslant \xi_{max.}$$

if we put

$$t := \frac{\xi - \xi_0}{\xi + \xi_0} \qquad \xi_0 = \sqrt{\xi_{min.} \xi_{max}} \tag{3.1.5}$$

so that $\log \xi = \log \xi_0 + \log \frac{1 + t}{1 - t}$ (3.1.6)

The range of t is then given by

$$|t| \leqslant t_{max} := \frac{\xi_{max} - \xi_o}{\xi_{max} + \xi_o} \tag{3.1.7}$$

If we apply this method to the range $\frac{1}{2} \leqslant \xi \leqslant 1$ (cf. (3.1.1) above), we obtain

$$\begin{aligned} \xi_o &= \sqrt{\tfrac{1}{2}} = .70710\ 67811\ 86547\ 5244 \\ \log_2 \xi &= 0.5 + \log \frac{1 + t}{1 - t} \\ t_{max} &= \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \simeq .17157\ 28752\ 5381 \end{aligned} \tag{3.1.8}$$

Thus t^2_{max} is just a little less than $3 \cdot 10^{-2}$ and each term of the power series (3.1.3) will add almost 2 more decimals; or a little more than 2 if the corresponding best-fit polynomial is used.

While the reduction (3.1.8) will be sufficient to allow for fairly short rational approximations, it is worthwhile to note that a further reduction is possible without introducing any new explicit operations (i.e. other than those for determining the proper range). For if we divide the interval $[\xi_{min}, \xi_{max}]$ into n subintervals:

$$k^{th} \text{ interval} = [\xi_{k-1,k}, \xi_{k,k+1}]$$

$$\xi_{min} = \xi_{o,1} < \xi_{1,2} < \cdots < \xi_{n-1,n} < \xi_{n,n+1} = \xi_{max} \quad (3.1.9)$$

and define $\xi_k := \sqrt{\xi_{k-1,k} \cdot \xi_{k,k+1}}$

then for each $\xi \in [\xi_{k-1,k}, \xi_{k,k+1}]$

we take

$$t := \frac{\xi - \xi_k}{\xi + \xi_k} \quad (3.1.10)$$

and hence have $|t| \leq t_k := \dfrac{\xi_k - \xi_{k,k-1}}{\xi_k + \xi_{k,k-1}}$

or $t_k = \dfrac{r_k - 1}{r_k + 1}$, $r_k := \sqrt{\xi_{k+1,k} / \xi_{k,k-1}}$

If the number of subintervals, n, is given then maximum $(t_1, t_2, \ldots t_n)$ is minimized by logarithmic subdivision, viz.

$$r_1 = r_2 = \ldots = r_n \quad (3.1.11)$$

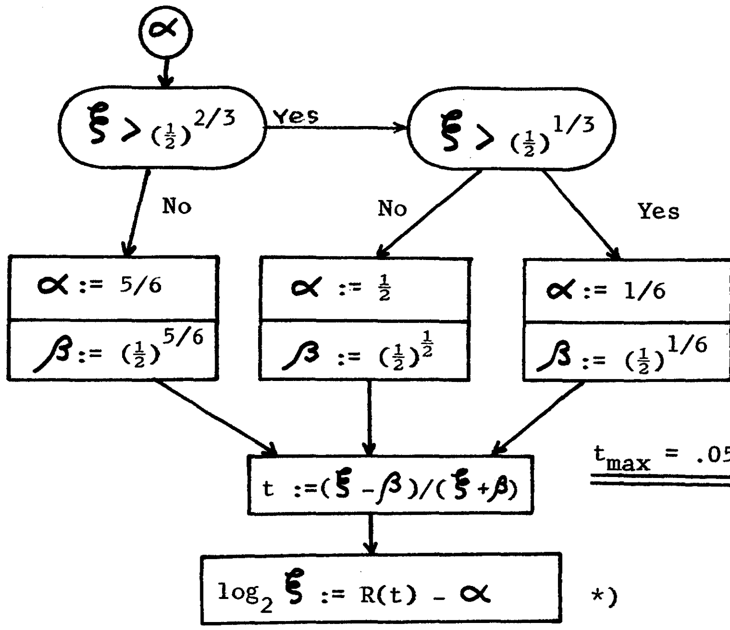
but linear subdivision, viz.

$$\xi_{k,k+1} = \xi_{min} + \frac{k}{n}\left(\xi_{max} - \xi_{min}\right) \quad (3.1.12)$$

may save enough time and/or storage to offset its lesser theoretical efficiency.

EXAMPLES:

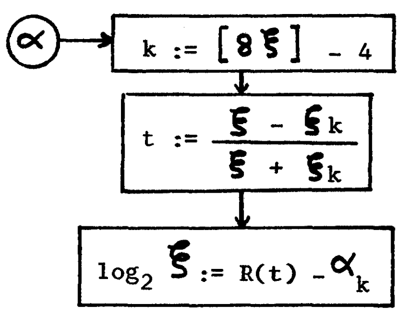
(i) Logarithmic Subdivision, $\min = \frac{1}{2}$, $\max = 1$, $n = 3$



$t_{max} = .05769\ 81098$

*)

(ii) Linear Subdivision, $\min = \frac{1}{2}$, $\max = 1$, $n = 4$



$[\ \ ]$ = "integer part of"

$\xi_o = \sqrt{5/16} = \sqrt{.3125}$

$\xi_1 = 11/16$; $\xi_2 = 13/16$; $\xi_3 = 15/16$

$-\alpha_k = \log_2 \xi_k$

*)

$t_{max} = \dfrac{\sqrt{5/4} - 1}{\sqrt{5/4} + 1} = .05572\ 80900$

In these two examples, t_{max} is not much different, though (ii) has n = 4, (i) only n = 3.

---

*) $R(t) \approx \log_2 \dfrac{1 + t}{1 - t}$ is a suitable rational approximation.

(3.2) Rational Approximations for $\ln \frac{1+t}{1-t}$

(i) Approximations of the form

$$\ln \frac{1+t}{1-t} \approx R(t) := t(a^* + \frac{b^*}{c + t^2}) \quad *) \qquad (3.2.1)$$

This simple approximation is not accurate enough unless $[\frac{1}{2}, 1]$ is further divided. Approximate maximum errors (absolute errors) are given below:

TYPE OF SUBDIV.	n	max $\frac{\xi_{k,k+1}}{\xi_{k-1,k}}$	λ
none	1	2	3.29×10^{-9}
logarithmic	2	2	2.61×10^{-11}
logarithmic	3	2 1/3	1.53×10^{-12}
linear	4	1.25	1.20×10^{-12}
logarithmic	4	2 1/4	2.04×10^{-13}
linear	8	1.125	1.13×10^{-14}
logarithmic	8	2 1/8	1.60×10^{-15}

TABLE (3.2.2)

For a standard floating point subroutine the maximum error should be below $7 \cdot 10^{-12}$, thus "logarithmic, n = 3" and "linear, n = 4" are suitable. For many special purposes, a routine which gives the logarithm in fixed point will also be very useful. The argument may be in fixed or floating representation, and a somewhat greater accuracy may then be required. We presently have computed the following coefficients of approximation.

---

*) For $\log_2 \frac{1+t}{1-t}$, divide the constants marked with an asterisk by

ln 2 = .69314 71805 59945 30941 72321 .... (cf. Tables Nat. Log. Vol. II, N.B.S. Appl. Math. Series #53)

$\max \frac{(\xi_{k,k+1})}{(\xi_{k-1,k})}$	t_{max}	a^*, b^*, c	λ
2	.17157 2875	a* = .89554 02099 560 b* = -1.82984 55434 565 c = -1.65677 85798 852	$3.29\ 10^{-9}$
$2^{1/2}$	.08642 7234	a* = .89055 57990 96268 b* = 1.84630 58864 29456 c = 1.66417 19172 70150	$2.61\ 10^{-11}$
$2^{1/3}$	.05769 8110 *)	a* = .88963 00669 363587 b* = -1.84938 33168 136899 c = -1.66555 60110 514012	$1.53\ 10^{-12}$
$2^{1/8}$	.02165 7463	a* = .88899 31487 3553390 b* = -1.85150 43597 8820645 c = -1.66651 02989 0598803	$1.6\ 10^{-15}$

TABLE (3.2.3)

For very small ranges, up to about $\xi_{max} / \xi_{min} = 1.2$, i.e., $t_{max} \approx .05$, the "telescoping procedure for continued fractions"#) can be used to compute a*, b* c with sufficient accuracy. For this particular case we obtain, with $\epsilon = t_{max}$:

$$\ln \frac{1+t}{1-t} \approx t\, \frac{P_0^* + P_1^*\, t^2}{q_0 + q_1\, t^2} \qquad \left[\begin{array}{l} P_0^* = 30 + \frac{3}{40}\, t^6_{max};\; P_1^* = -8 - \frac{18}{5}\, t^2_{max} \\ q_0 = 15;\quad q_1 = -9 - \frac{9}{5}\, t^2_{max} + \frac{3}{10}\, t^4_{max} \end{array} \right] \qquad (3.2.4)$$

from which the corresponding expression of the form (3.2.1) can easily be derived.

#) cf. copy of my paper on this subject: "Methods for Fitting Rational Approximations", J.A.C.M., 1960.

*) May also be used for n = 4, linear, as long as the exact values (yielding $\lambda = 1.45 \cdot 10^{-12}$) are not known.

(ii) Approximations of the Form

$$\ln \frac{1+t}{1-t} \approx R(t) := t\left[a^* + t^2\left(b^* + \frac{c^*}{d+t^2}\right)\right] \quad *) \qquad (3.2.5)$$

This approximation yields nearly three decimals more than (3.2.1) - for the ranges given below, but it also requires one additional constant, one more addition and one more multiplication. This may not be too high a price if the subdivision of the interval $\left[\frac{1}{2}, 1\right]$ can thereby be avoided; however, the error for the full range is approximately 10^{-11}, which is slightly above the basic round-off of $7 \cdot 10^{-12}$. Further error estimates are given below:

TYPE OF SUBDIV.	n	$\max \frac{\xi_{k,\,k+1}}{\xi_{k-1,\,k}}$	λ (appr.)
none	1	2	10^{-11}
logarithmic	2	$2^{\frac{1}{2}}$	$2 \cdot 10^{-14}$
logarithmic	3	$2^{1/3}$	$5 \cdot 10^{-16}$
linear	4	1.25	$4 \cdot 10^{-16}$
logarithmic	4	$2^{\frac{1}{4}}$	$4 \cdot 10^{-17}$

TABLE (3.2.6)

The constants a*, b*, c* and d for the full range have been computed for this report:

$\xi_{max} / \xi_{min} = 2$

a* = 1.99999 99994 91255
b* = .10907 88905 02997
c* = - .77731 40010 05492
d = - 1.39406 51451 76107

$t_{max} = .17157288$

maximum abs. error $\lambda = 1 \cdot 10^{-11}$

TABLE (3.2.7)

---

*) For $\log_2 \frac{1+t}{1-t}$ divide the constants marked with an asterisk by ln 2 = .69314 71805 59945 30941 72321 ...

(iii) Approximations of the Form

$$\ln \frac{1+t}{1-t} \approx R(t) = \frac{t}{a' + t^2\left(b' + \frac{c'}{d + t^2}\right)} \quad *) \qquad (3.2.8)$$

This approximation is only slightly more accurate, the error being about 64% of that of the previous approximation, (3.2.5). The number of constants is the same, but one multiplication has been replaced by a division. Furthermore, this form (3.2.8) is somewhat more susceptible to round-off errors (due to the finite word-length) during evaluation than the form (3.2.5). However, this need not bother us if the evaluation is done in fixed point (carefully scaled) with correct round-off to a floating point mantissa at the end.

For the full range $[\frac{1}{2}, 1]$ the maximum absolute error will be approximately $6.5 \cdot 10^{-12}$ (Note: The constants a', b', c' and d are not included in this report.)

For the half ranges, $[\frac{1}{2}, \sqrt{\frac{1}{2}}]$ and $[\sqrt{\frac{1}{2}}, 1]$, the maximum absolute error will be approx. $1.3 \cdot 10^{-14}$, but this approximation cannot be recommended for a full precision fixed-point routine since the round-off error will be bigger than in (3.2.5), and (3.2.9) will be faster.

---

*) For $\log_2 \frac{1+t}{1-t}$, multiply the constants a', b', c' by $\ln 2 = .69314\ 71805\ 59945\ 30941\ 72321 \cdots$.

(iv) Approximation of the Form

$$\ln \frac{1 + t}{1 - t} \approx R(t) = t \left[a^* + \frac{b^*}{c + t^2 + \frac{d}{e + t^2}} \right] \text{*)} \qquad (3.2.9)$$

This approximation is much more accurate than (3.2.8); it requires one more constant but only one more addition; the number of multiplications (1) and divisions (2) is the same. This approximation can be used for a full-precision fixed-point subroutine without subdividing the interval $[\frac{1}{2}, 1]$. The constants are:

for

$\xi_{max}/\xi_{min} = 2$, $t_{max} = .17157288$, $\lambda = 1.18\ 10^{-14}$

TABLE (3.2.10)

a^*	=	.57314 62238 34578
b^*	=	-3.83907 86035 23797
c	=	-3.08667 66195 74836
d	=	- .61016 03452 67418
e	=	-1.54047 22733 27729

Which of the approximations given above are most suitable for Control Data subroutines will depend not only on certain details of coding and machine characteristics, but also on the relative merits of saving time or storage space and on the range for which a fixed point logarithm may be used, i.e., on the number of bits available for the fractional part of a fixed point logarithm after suitable scaling.

---

*) For $\log_2 \frac{1 + t}{1 - t}$ divide the constants marked with an asterisk by ln 2 = .69314 71805 59945 30941 72321 ...

4. APPROXIMATIONS FOR THE ARCTAN Z

(4.1) Reduction of the Range

The addition theorem for tan $(\phi + \psi)$, viz.

$$\tan(\phi + \psi) = \frac{\tan\phi + \tan\psi}{1 - \tan\phi \tan\psi} \quad (4.1.1)$$

can be used to reduce the range for which arctan X or arctan (X/Y) must be computed, if we store, in the memory of the machine, a table of "key values" Z_k, ψ_k,

$$Z_k := \tan \psi_k$$
$$\psi_k = \arctan Z_k \quad (4.1.2)$$

The subroutine will first find, for each argument Z, the table entry Z_k which is "nearest" to Z in the sense that $|\arctan Z - \psi_k|$ is minimized. After that the algorithm runs as follows:

$$t := \frac{Z - Z_k}{1 + ZZ_k} \quad {*)} \qquad (= \tan\phi) \quad (4.1.3)$$
$$\phi :\cong \arctan t \quad \text{(approximation)}$$
$$\arctan Z := \psi_k + \phi$$

While this algorithm permits a very drastic reduction of the range its cost in time and storage is considerable. Time is lost for finding the best (ψ_k, Z_k), and for computing t (requiring one each of the four basic operations).

---

*) To compute arctan X/Y, use

$$t := \frac{X - YZ_k}{Y + XZ_k} \quad (4.1.4)$$

Storage space is needed at least for the Z_k, while the ψ_k may be equidistant.

Conclusion: With a small table, too much time is lost and not enough is gained by the moderate reduction of the range; a large table will save some time but cost more memory space than desirable for a general purpose subroutine.

Special Values: The formula for t is, of course, very simple for $Z_k = 0$ ($t = Z$) and for $Z_k \to \infty$ ($t = \frac{1}{Z}$), but also for $Z_k = \pm 1$, ($t = (Z \mp 1)/(1 \pm Z)$). With these four values, we obtain the following short table:

RANGE OF Z	Z_k	ψ_k	t
$Z < -(1 + \sqrt{2})$	$-\infty$	$-\pi/2$	$-\frac{1}{Z}$
$-(1 + \sqrt{2}) < Z < -(\sqrt{2} - 1)$	-1	$-\pi/4$	$\frac{1 + Z}{1 - Z}$
$\|Z\| < \sqrt{2} - 1$	0	0	Z
$\sqrt{2} - 1 < Z < 1 + \sqrt{2}$	+1	$+\pi/4$	$\frac{Z - 1}{Z + 1}$
$Z > 1 + \sqrt{2}$	$+\infty$	$+\pi/2$	$-\frac{1}{Z}$

*) (4.1.5)

The first and last ranges can be joined if it is not required that

$$-\pi/2 < \arctan Z < +\pi/2$$

This method reduces the range to $|t| \leq \sqrt{2} - 1$

*) It is true that t can also be computed very easily for arctan X/Y, since, if $Z = X/Y$, $\frac{1}{Z} = \frac{Y}{X}$; $\frac{1 + Z}{1 - Z} = \frac{Y + X}{Y - X}$ and $\frac{Z - 1}{Z + 1} = \frac{X - Y}{X + Y}$
but how can we determine the range directly from X & Y, without computing Z?

(4.2) APPROXIMATIONS FOR ARCTAN t, $|t| \leq \sqrt{2} - 1$

We have derived the following approximation from the continued fraction for arctangent of t

$$R_6(t) \approx \arctan t \quad \text{if}$$

$$R_6(t) := t\left[d_1 + \cfrac{e_1}{t^2 + d_2 + \cfrac{e_2}{t^2 + d_3 + \cfrac{e_3}{t^2 + d_4}}}\right] \tag{4.2.1}$$

with

$$\left.\begin{aligned}
d_1 &= 0.20131\ 20564\ 40625\ 303 \\
e_1 &= 3.11385\ 00604\ 57103\ 14 \\
d_2 &= 5.40622\ 85377\ 62366\ 96 \\
e_2 &= -3.92831\ 57487\ 32049\ 88 \\
d_3 &= 2.71829\ 04240\ 10983\ 87 \\
e_3 &= -0.15058\ 39379\ 13062\ 15 \\
d_4 &= 1.33875\ 95795\ 46815\ 11
\end{aligned}\right] \tag{4.2.2}$$

The maximum relative error λ,

$$\lambda_6 := \max_{|t| \leq \sqrt{2}-1} \left| \log \frac{R_6(t)}{\arctan t} \right| = 2.84 \cdot 10^{-14} \tag{4.2.3}$$

is much smaller than necessary for a good floating point routine; it is actually good enough for a fixed point routine, since the absolute error, $\lambda_{6\ abs} \approx 3 \cdot 2^{-48}$

$$\lambda_{6\ abs} = \max_{|t| \leq \sqrt{2}-1} \left| R_6(t) - \arctan t \right| = 1.11 \cdot 10^{-14} \tag{4.2.4}$$

For a floating point routine the somewhat simpler approximation R_5 (t) may be used:

$$R_5(t) := t\left[d_o + t^2\left(d_1 + \cfrac{e_1}{t^2 + d_2 + \cfrac{e_2}{t^2 + d_3}}\right)\right]$$

$(\lambda = 3.9\cdot 10^{-12})$

where

$$\left.\begin{aligned}
d_o &= 0.99999\ 99999\ 96107\\
d_1 &= -0.01558\ 53710\ 18178\\
e_1 &= -0.58531\ 51350\ 71831\\
d_2 &= 2.10055\ 40871\ 65198\\
e_2 &= -0.41900\ 30022\ 82544\\
d_3 &= 1.62102\ 38336\ 34443
\end{aligned}\right] \qquad (4.2.5)$$

5. APPROXIMATIONS FOR TAN X

(5.1) REDUCTION OF THE RANGE

The basic range for the tangent is given by the periodicity of tan X.

$$\tan (X \pm \pi) \equiv \tan X \tag{5.1.1}$$

Therefore, the basic range is

$$\alpha - \pi/2 \leq X \leq \alpha + \pi/2 \tag{5.1.2}$$

where α may be chosen to be zero.

Further reductions can easily be achieved by the relation (4.1.1) of which

$$\tan (\phi \pm \pi/2) = -1/\tan \phi \tag{5.1.3}$$

is a special case. Another important special case is

$\psi = \pm \pi/4, \quad \tan \psi = \pm 1$, thus

$$\tan (\phi \pm \pi/4) = \frac{\tan \phi \pm 1}{1 \mp \tan \phi} \tag{5.1.4}$$

(5.1.3) cuts the basic range to $\underline{\underline{|X| \leq \pi/4}}$; with (5.1.4), we get down to $\underline{\underline{|X| \leq \pi/8}}$ *)

A code for the reduction of the range can be made efficient only if we introduce an auxiliary variable $W = \frac{X}{\pi} \cdot 2^k$. We have arbitrarily chosen k = 2, thus

$$W := \frac{4}{\pi} X \tag{5.1.5}$$

Let us write $\quad t(W) := \tan \frac{\pi W}{4} = \tan X \qquad (5.1.6)$

then

$$t(W \pm 4) = t(W) \tag{5.1.7}$$

$$t(W \pm 2) = -1/t(W) \tag{5.1.8}$$

$$t(W \pm 1) = \frac{t(W) \pm 1}{1 \mp t(W)} \tag{5.1.9}$$

---

*) By help of a table of key values and using (4.1.1), the range can be further reduced, saving a little time at great cost in storage.

(5.2) BASIC FORM OF RATIONAL APPROXIMATION FOR TAN X

Since tan X and t(W) are odd functions, we may write

$$t(W) = W\cdot T\,(W^2) = \frac{W}{S(W^2)} \tag{5.2.1}$$

The question is whether $T(W^2)$ or $S(W^2)$ should be used as an auxiliary function. Assuming that T and S could be computed equally fast (with comparable accuracy), the first form will be faster for the basic range since a multiplication takes less time than a division. However, if one of the relations (5.1.8) or (5.1.9) must be used, then the second form is much faster since

$$t(W \pm 2) = -\frac{1}{t(W)} = \frac{1}{W\cdot T} = \frac{S}{W} \tag{5.2.2}$$

and

$$t(W \pm 1) = \frac{t(W) \pm 1}{1 \mp t(W)} = \frac{WT \pm 1}{1 \mp WT} = \frac{W \pm S}{S \mp W} \tag{5.2.3}$$

Which method will be faster on the average? This depends on the frequency of arguments being in the basic range ($|X| \leq \pi/4$ or $\pi/8$) as opposed to those in the ranges requiring reduction by (5.2.2) or (5.2.3). If the distribution is uniform, then the second form is faster. *) The use of $S(W^2)$ also reduces the maximum time for the subroutine. Therefore, $S(W^2)$ will be recommended for a general purpose subroutine.

---

*) While this is true for both ranges ($|X| \leq \pi/4$, $|X| \leq \pi/8$), the difference is bigger if the smaller range, and thus (5.2.3), is used.

(5.3) RATIONAL APPROXIMATIONS FOR $S(W^2)$

Since

$$\tan X = \frac{X|}{|1} - \frac{X^2|}{|3} - \frac{X^2|}{|5} - \frac{X^2|}{|7} - \frac{X^2|}{|9} - \cdots \qquad (5.3.1)$$

$S(W^2)$ can be expressed by the continued fraction

$$S(W^2) = \frac{W}{t(W)} = W/\tan\frac{\pi W}{4}$$

$$= \frac{4}{\pi} - \frac{\frac{\pi}{4} W^2|}{|3} - \frac{\frac{\pi^2}{16} W^2|}{|5} - \frac{\frac{\pi^2}{16} W^2|}{|7} - \cdots \qquad (5.3.2)$$

In the formulae given below, this expression has been modified in two ways:

(i) The coefficients have been modified for best fit for the respective interval, i.e. so that

$$\lambda_{rel.} = \max \left| \ln \frac{S^*_n(W^2)}{S(W^2)} \right| \qquad (5.3.3)$$

(= the relative error) is minimized.

(ii) Simple algebraic transformations have been used to minimize the time required for evaluating the respective expression on the Control Data 1604.

If n is the "degree" of the approximation, viz.

$$S^*_n(W^2) = C_o + \frac{C_1 W^2|}{|1} + \cdots + \frac{C_n W^2|}{|1} \qquad (5.3.4)$$

We obtain the following short table of maximum relative errors ($\lambda_{rel.}$):

n	$\lvert X \rvert \leq \pi/4$	$\lvert X \rvert \leq \pi/8$
3	$1.42 \cdot 10^{-8}$	$4.69 \cdot 10^{-11}$
4	$2.21 \cdot 10^{-11}$	$1.83 \cdot 10^{-14}$
5	$2.38 \cdot 10^{-14}$	$4.92 \cdot 10^{-18}$

TABLE (5.3.5)

(i) APPROXIMATIONS FOR $|X| \leqslant \pi/4$

REDUCTION OF THE RANGE:

$$X \left(\frac{2}{\pi}\right) =: 2i + k + v$$

where i and k are integers and $\left[\begin{array}{ll} |k+v| & \leqslant 1 \\ |v| & \leqslant \frac{1}{2} \end{array}\right]$ ALGOR (5.3.6)

for $k = \pm 1$; tan X := − S/W

for k = 0 ; tan X := W/S

where either

$$S := S_4^*(v^2) = a + \cfrac{b}{c + v^2 + \cfrac{d}{e + v^2}} \quad *)$$

$(\lambda = 2.21 \cdot 10^{-11})$

a = 9.45815 57617 25496

b = 290.32031 00841 78635

c = −37.33612 85498 26952 (5.3.7)

d = −20.54475 60663 69045

e = −4.64212 22417 14098

or

$$S := S_5^*(v^2) = a + v^2 \left[b + \cfrac{c}{d + v^2 + \cfrac{e}{f + v^2}} \right]$$

$\lambda = 2.38 \times 10^{-14}$

a = .63661 97723 67596

b = −.07531 94869 91705 (5.3.8)

c = 3.88560 57227 68290

d = −14.87026 86251 97861

e = −57.81869 13873 68667

f = −9.32191 89536 46030

---

*) About 4 to 5 bits ($1\frac{1}{2}$ decimals) may be lost by amplified round-off with this slightly unstable approximation. It is recommended to evaluate S_4^* in fixed point (48 bits) with proper scaling.

(ii) APPROXIMATIONS FOR $|X| \leqslant \pi/8$

REDUCTION OF THE RANGE: ALGOR. (5.3.9)

$X\left(\frac{4}{\pi}\right) =: 4i + k + W$

Where i and k are integers and $\left[\begin{array}{r} |k + W| \leqslant 2 \\ |W| \leqslant \frac{1}{2} \end{array}\right]$

for $k = \pm 2$; $\tan X := -S/W$

for $k = +1$; $\tan X := \dfrac{W + S}{S - W}$

for $k = -1$; $\tan X := \dfrac{W - S}{S + W}$

for $k = 0$; $\tan X := W/S$

Where either: $S := S_3^*(W^2) = a + W^2\left(b + \dfrac{c}{d + W^2}\right)$ (5.3.10)

With ($\lambda = 4.69 \cdot 10^{-11}$)

a = 1.27323 95447 94842

b = - .07881 15321 78328

c = 3.11023 62587 99796

d = -16.99695 38195 49826

Or $S := S_4^*(W^2) = a + \dfrac{b}{c + W^2 + \dfrac{d}{e + W^2}}$ *)   (5.3.11)

With ($\lambda = 1.83 \cdot 10^{-14}$)

a = 19.05374 90250 96548

b = 2368.80216 75614 4904

c = -151.08047 87151 32145

d = -331.56482 92387 31320

e = - 18.56899 78562 14913

---

*) About 4 to 5 bits ($1\frac{1}{2}$ decimals) may be lost by amplified round-off with this slightly unstable approximation. It is recommended to evaluate S_4^* in fixed point (48 bits) with proper scaling.

6. APPROXIMATIONS FOR SIN X AND COS X

(6.1) REDUCTION OF THE RANGE

The first and basic reduction of the range of the independent variable is brought about by the periodicity:

$$\left.\begin{array}{l}\sin X = \sin (X + 2k\pi)\\ \cos X = \cos (X + 2k\pi)\end{array}\right. \quad k = \pm 1, \pm 2, \pm 3, \ldots \tag{6.1.1}$$

and the basic relations

$$\left.\begin{array}{l}\sin (X \pm \pi) = -\sin X\\ \cos (X \pm \pi) = -\cos X\end{array}\right] \tag{6.1.2}$$

These reduce the range for which rational approximations for sin X and cos X must be found to

$$-\pi/2 \leq X \leq +\pi/2 \tag{6.1.3}$$

Further reductions are possible, but most of them do not pay:

(i) Since $\sin(-X) = -\sin X$, $\cos(-X) = \cos X$, we could reduce the range to $0 \leq X \leq \pi/2$. But since this range is not symmetric, additional terms would appear in a best-fit approximation (viz. even powers for sin X, odd powers for cos X). Nothing would be gained, neither in speed nor in storage space.

(ii) Formulae such as $\cos X = 2\cos^2(X/2) - 1$ or $\sin X = \sin\frac{X}{3}\left(3 - 4\sin^2\frac{X}{3}\right)$, etc. could also be used, but the evaluation of these formulae takes more time than can be saved by the respective reduction of the range, except for extremely high precision (double or triple precision).

(iii) Since the continued fraction and rational approximations for tan X are so good, we may be tempted to compute sin X and cos X from

$$t := \tan\frac{X}{2} \qquad \left.\begin{aligned} \sin X &= \frac{2t}{1+t^2} \\ \cos X &= \frac{1-t^2}{1+t^2} \end{aligned}\right] \tag{6.1.4}$$

Here again, the evaluation of $2t/(1 + t^2)$ or $(1 - t^2) / (1 + t^2)$ takes more time than can be saved.

(iv) If we store a table of key values $S_k = \sin X_k$ and $C_k = \cos X_k$, then sin X and cos X may be computed from

$$X = X_k + \xi \qquad \left.\begin{aligned} \sin X &= S_k \cos\xi + C_k \sin\xi \\ \cos X &= C_k \cos\xi - S_k \sin\xi \end{aligned}\right] \tag{6.1.5}$$

This requires two additional multiplications and a double table look up; in addition, <u>both</u> <u>$\cos\xi$</u> <u>and</u> <u>$\sin\xi$</u> <u>must</u> <u>be</u> <u>approximated</u>. This method does not pay off unless the range of ξ is made extremely small -- but this means a long table of many key values (perhaps 1000 or so).

(v) So far, we have not yet made use of the relation

$$\sin X = \cos(\pi/2 - X) \tag{6.1.6}$$

This can be used to obviate the need for one of the subroutines, *) but

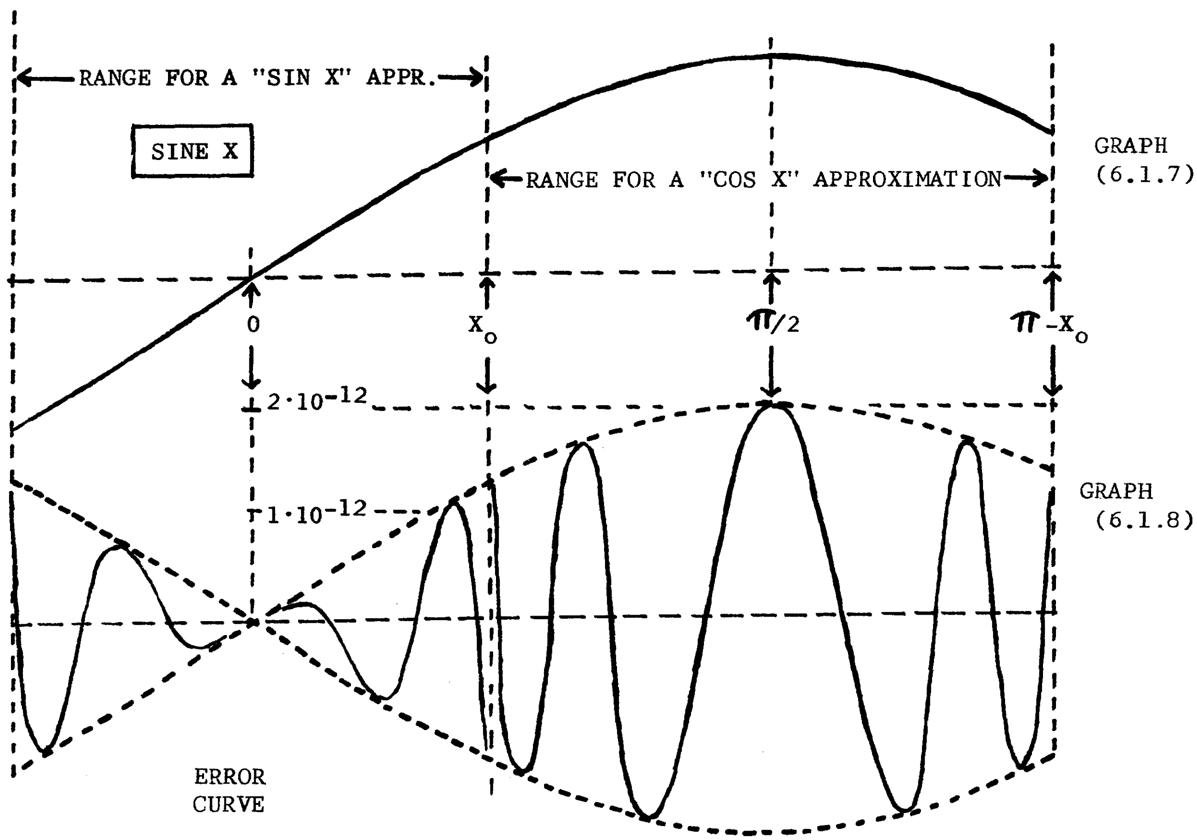
---

*) Since sin X should come out as zero for X = 0, and good relative accuracy is desirable even if $|X| \ll 1$, for sin X, the cos X subroutine may be dropped, but <u>the sin X - subroutine should be retained</u>.

it may also be used to reduce the range by taking "the other function" if $|X| > X_o$

where $0 < X_o < \pi/2$

may be determined in such a way that the maximum error of two given forms of best-fit approximations for sin X & cos X are just equally big.



The sketch above illustrates this situation, assuming that the <u>relative</u> error has been minimized.

<u>This is the only reduction as far as I can see, which is worth while</u>.

(6.2) MATCHED RATIONAL APPROXIMATIONS FOR
SIN X, $|X| < X_0$, & COS X, $|X| < \pi/2 - X_0$

(i) A pair of approximations of the form

$$\left.\begin{aligned} \sin X &:\doteq X\left(S_1 + \cfrac{S_2}{X^2 + S_3 + \cfrac{S_4}{X^2 + C_5}}\right) \\ \cos X &:= C_1 + \cfrac{C_2}{X^2 + C_3 + \cfrac{C_4}{X^2 + C_5}} \end{aligned}\right] \quad (6.2.1)$$

will yield an error of approximately 2×10^{-11} *)

The matching point is approximately $X_0 \approx .90$. Since 2×10^{-11} is too big even for (full precision) floating point, the coefficients S_1 through C_5 have not been computed yet. They can be furnished on request.

(ii) A slightly better pair of approximations is

$$\left.\begin{aligned} \sin X &:= X\left(S_1 + X^2\left(S_2 + X^2\left(S_3 + \frac{S_4}{X^2 + S_5}\right)\right)\right) \\ \cos X &:= C_1 + X^2\left(C_2 + X^2\left(C_3 + \frac{C_4}{X^2 + C_5}\right)\right) \end{aligned}\right] \quad (6.2.2)$$

The relative error is about $\lambda = 8 \times 10^{-12} \approx 1.1 \times 2^{-37}$ which may be just acceptable for a 1604 floating point subroutine. The matching point is between .88 and .89. The coefficients can be computed fairly easily by a relatively simple iteration and will be furnished if requested.

NOTE: The numerical stability of (6.22) is much better than that of (6.2.1). (6.2.2) can be evaluated in floating point.

---

*) All errors quoted are relative, i.e.

$$\lambda = \max \left| \ln \frac{\text{APPROX.}}{\text{FUNCTION}} \right|$$

(iii) The following pair of approximations will yield 12 significant digits:

$$\left.\begin{aligned} \sin X &:= X\left(S_1 + \cfrac{S_2}{X^2 + S_3 + \cfrac{S_4}{X^2 + S_5}}\right) \\ \cos X &:= C_1 + X^2\left(C_2 + \cfrac{C_3}{X^2 + C_4 + \cfrac{C_5}{X^2 + C_6}}\right) \end{aligned}\right] \qquad (6.2.3)$$

relative error: $\lambda \cong 4.56 \cdot 10^{-13}$

matching point: $X_o \cong .6271$

COEFFICIENTS: *)

S_1 =	7.23084 68962 44279	C_1 =	.99999 99999 99545
S_2 =	814.80758 58531 22316	C_2 =	- 1.67714 58152 33633
S_3 =	55.40962 23983 32114	C_3 =	271.20667 68780 46237
S_4 =	1262.62414 34758 4584	C_4 =	80.85518 72908 64723
S_5 =	16.75449 20850 08428	C_5 =	2442.54254 69501 6347
		C_6 =	16.33389 75777 12791

(iv) Since $|\sin X| \leqslant 1, |\cos X| \leqslant 1$, a fixed point sin X cos X subroutine is also feasible, approximately 14 digits will be needed for full fixed point accuracy. The following pair of approximations may be used:

$$\left.\begin{aligned} \sin X &:= X\left[S_1 + X^2\left(S_2 + \cfrac{S_3}{X^2 + S_4 + \cfrac{S_5}{X^2 + S_6}}\right)\right] \\ \cos X &:= C_1 + X^2\left(C_2 + \cfrac{C_3}{X^2 + C_4 + \cfrac{C_5}{X^2 + C_6}}\right) \end{aligned}\right] \qquad (6.2.4)$$

$\lambda = 9.5 \cdot 10^{-15}$, $X_o = .8798$

---

*) cf. "NOTE" on p. 42

The coefficients for (6.2.4) are:

$S_1 = .99999\ 99999\ 99990\ 520$	$C_1 = .99999\ 99999\ 99990\ 455$
$S_2 = -.32590\ 23686\ 32526\ 89$	$C_2 = 1.70422\ 86567\ 42058\ 33$
$S_3 = 71.63509\ 21318\ 01290\ 7$	$C_3 = 276.53438\ 35490\ 61398$
$S_4 = 113.84748\ 44349\ 29748$	$C_4 = 81.38352\ 57153\ 53261\ 8$
$S_5 = 4600.47709\ 02433\ 9799$	$C_5 = 2456.97667\ 28922\ 5259$
$S_6 = 13.69104\ 85436\ 09095\ 3$	$C_6 = 16.57290\ 96384\ 87355\ 5$

The numerical stability of the formulae (6.2.4) is fair. Even with careful scaling the round-off error may be bigger than the truncation error. The <u>next</u> <u>approximation</u> will reduce both at a moderate increase in computing time.

(v) The following pair looks fine for a high-precision fixed-point subroutine:

$$\left.\begin{aligned} \sin X &:= X\left(S_1 + X^2\left(S_2 + X^2\left(S_3 + X^2\left(S_4 + \frac{S_5}{X^2 + S_6}\right)\right)\right)\right) \\ \cos X &:= C_1 + X^2\left(C_2 + X^2\left(C_3 + X^2\left(C_4 + \frac{C_5}{X^2 + C_6}\right)\right)\right) \end{aligned}\right] \quad (6.2.5)$$

$$\lambda \cong 8.1 \times 10^{-15}, \qquad X_o \approx .885$$

Since sin X and cos X must obviously be scaled by a factor of 1/2 in order to avoid overflow, $\lambda_{rel} = 8.1 \times 10^{-15}$ means that the truncation error is only slightly in excess of 1 unit of the last binary. Since (6.2.5) is very stable, the sum of round-off plus truncation error should be less than 2 or 3 units of the last binary for most values of X.

(6.3) RATIONAL APPROXIMATIONS FOR SIN X, $|X| \leqslant \pi/2$

Approximations to sin X for the full range, $-\pi/2 \leqslant X \leqslant +\pi/2$ require one division or multiplication more than those in (6.2). The coefficients have not yet been computed, except for the first approximation, the accuracy of which is not sufficient for a full-precision floating point subroutine. The error estimates for the other approximations are believed to be correct within about $\pm$ 10% of the values given below. (Please ask for coefficients if interested.)

(i) APPROXIMATION OF THE FORM SIN X $\approx X \cdot P_5(X^2)$

$$\sin X := X\,(S_1 + X^2\,(S_2 + X^2\,(S_3 + X^2\,(S_4 + X^2\,(S_5 + X^2\,S_6)))))$$

$\lambda_{rel} = 2.1 \times 10^{-11}$

S_1 = +.99999 99999 79082 *)

S_2 = -.16666 66660 92171 (6.3.1)

S_3 = +.00833 33307 30723

S_4 = -.00019 84083 38222

S_5 = +.00000 27524 01177

S_6 = -.00000 00238 68930

(ii) APPROXIMATION OF THE FORM SIN X = X $P_4(X^2)/Q_1(X^2)$

$$\sin X := X(S_1 + X^2\,(S_2 + X^2\,(S_3 + X^2\,(S_4 + \frac{S_5}{X^2 + S_6})))) \qquad (6.3.2)$$

$\lambda_{rel} = 1.0 \times 10^{-11}$

This approximation is at least twice as accurate as (6.3.1) and can be computed faster, since 2 multiplications have been replaced by 1 division. Numerical stability is very good.

---

*) cf. "NOTE" p. 42

(iii) APPROXIMATION OF THE FORM $\sin X \cong X \cdot P_3(X^2)/Q_2(X^2)$

$$\sin X := X\left(S_1 + X^2\left(S_2 + \frac{S_3}{X^2 + S_4 + \frac{S_5}{X^2 + S_6}}\right)\right) \qquad (6.3.3)$$

$\lambda_{rel} \cong 1.1 \times 10^{-11}$

Two more multiplications have been replaced by one division, saving time but losing about 10% in accuracy. Numerical stability is not as good as in (6.3.2); therefore, a well-scaled fixed-point evaluation is mandatory!

We now proceed to the next degree of approximation which will yield about 8 additional bits:

(iv) APPROXIMATION OF THE FORM $\sin X = X\, P_6(X^2)$

$$\sin X := X\left(S_1 + X^2\left(S_2 + X^2\left(S_3 + X^2\left(S_4 + X^2\left(S_5 + X^2\left(S_6 + S_7 X^2\right)\right)\right)\right)\right)\right) \qquad (6.3.4)$$

$\lambda \cong 6.2 \cdot 10^{-14}$ *)

(v) APPROXIMATION OF THE FORM $\sin X = X \cdot P_5(X^2)/Q_1(X^2)$

$$\sin X := X\left(S_1 + X^2\left(S_2 + X^2\left(S_3 + X^2\left(S_4 + X^2\left(S_5 + \frac{S_6}{X^2 + S_7}\right)\right)\right)\right)\right) \qquad (6.3.5)$$

$\lambda \cong 2.3 \cdot 10^{-14}$ *)

(vi) APPROXIMATION OF THE FORM $\sin X = X \cdot P_4(X^2)/Q_2(X^2)$

$$\sin X := X\left(S_1 + X^2\left(S_2 + X^2\left(S_3 + \frac{S_4}{X^2 + S_5 + \frac{S_6}{X^2 + S_7}}\right)\right)\right) \qquad (6.3.6)$$

$\lambda \cong 2.3 \cdot 10^{-14}$ *)

---

*) Numerical stability: Good for (6.3.4) through (6.3.6)

(6.4) SUMMARY AND NOTE

In my opinion, the following approximations deserve prime consideration:

FLOATING POINT:	Almost full accuracy	(6.2.2)
	Full accuracy	(6.2.3)
FIXED POINT:	Full accuracy	(6.2.5)

The approximations in (6.3) take more time and are a little less accurate.

NOTE:

It may be convenient to introduce, in the beginning of a sin & cos subroutine, an auxiliary variable

$$v := X \cdot \frac{2}{\pi}$$

(cf. ALGOR. 5.3.6), so that rational approximations not for sin X, but for $\sin \frac{v\pi}{2}$ are needed. The coefficients of such approximations can easily be found from those for sin X or cos X. An example is given below:

Substitute $\frac{v\pi}{2}$ for X in, say, (6.2.3)

$$\sin \frac{v\pi}{2} = \frac{v\pi}{2}\left(S_1 + \frac{S_2}{(v\pi/2)^2 + S_3 + \frac{S_4}{(v\pi/2)^2 + S_5}}\right)$$

$$= v\left(\sigma_1 + \frac{\sigma_2}{v^2 + \sigma_3 + \frac{\sigma_4}{v^2 + \sigma_5}}\right)$$

$$\sigma_1 = \frac{\pi}{2} S_1 \qquad \sigma_3 = \left(\frac{2}{\pi}\right)^2 S_3 \qquad \sigma_5 = \left(\frac{2}{\pi}\right)^2 S_5$$

$$\sigma_2 = \frac{2}{\pi} S_2 \qquad \sigma_4 = \left(\frac{2}{\pi}\right)^4 S_4$$

FINALE PRESTO

I should have liked to discuss more functions in this report but serious time limitations have prevented me from doing so.

We have some theoretical and numerical results for

$$\int_0^X e^{-t^2} dt \quad \text{and} \quad \int_0^X \frac{e^{-t}}{t} dt$$

in particular for $X \to \infty$ (whence $\int_X^\infty e^{-t^2} dt$ and $\int_X^\infty \frac{e^{-t}}{t} dt$).

A few simple functions, such as tanh X, ln cosh X, $\frac{e^X - 1}{X}$ and other functions for which either a well convergent power series or a well convergent continued fraction is known, can readily be treated, i.e., coefficients of suitable best-fit approximations can be obtained with our codes as soon as we find time for punching and running.

Some other functions, such as $\Gamma(X)$ for real values of X, have been studied and more work will be needed to make them ready for machine computation of the coefficients for best-fit approximations.

This semi-formal report has not been checked (for style, spelling and mathematical and numerical accuracy) as carefully as we would have checked a formal publication. Every comment and correction, and in particular reports on numerical checking of approximations given herein will be greatly appreciated.

ACKNOWLEDGEMENTS

Mr. Charles MESZTENYI has greatly contributed to all phases of the work reported here, viz., the analysis of special functions, planning and coding for the IBM 650 and running these codes for a great number of approximations (many more than listed in this report).

Dr. Christoph WITZGALL has also contributed to all phases, though he has joined our Project only one year ago.

Mrs. Birgitta NORELIUS-ASPLUND also joined us just a year ago but has taken a great burden of routine coding and running. Her accurate and reliable work has been most valuable.

THIS PROJECT AT PRINCETON UNIVERSITY

Was supported by the Bureau of Ships and its Applied Math. Lab., David Taylor Model Basin, under Contract Nonv 2406(00). It will terminate 31 JAN 1960.

Dr. WITZGALL will stay with Princeton University's Mathematics Dept. working on another project (Linear Programming) with Professor Tucker.

Mrs. NORELIUS will eventually return to Sweden, Mr. MESZTENYI and myself will go to Syracuse University where I hope to continue similar work.

PRINCETON
15 JAN 1960

Hans J. Maehly

Address all future correspondence to:

HANS J. MAEHLY
MATHEMATICS DEPT.
SYRACUSE UNIVERSITY
SYRACUSE 10, NEW YORK

CONTROL DATA CORPORATION

501 Park Avenue, Minneapolis 15, Minnesota