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A Note on the Resonant Modes and Spatial Coherency of a Fabry-Perot Maser Interferometer

Introduction

The resonant modes and spatial coherency obtaining in a Fabry-Perot maser interferometer have been studied by Fox and Li,[1] and Wolf,[2] and others. Fox and Li used an iteration method which describes transmission through an array of perfectly absorbing screens of appropriate aperture to show that after many transmissions a state is reached in which the relative field distribution does not vary from transit to transit. The steady state field distribution is regarded as a normal mode. Wolf showed that with the same periodic structure an incoherent beam becomes completely spatially coherent after a sufficient number of transmissions.

In this communication we establish mode selection and complete spatial coherency conditions for an illuminated Fabry-Perot cavity, both with and without an internal active medium. It is shown that after a sufficient number of reflections the output beam contains a single mode only if the cavity contains an active medium. In the case of a passive medium, however, it is not possible to obtain a single mode output if the input contains more than one mode, although the ratio of the energy of the mode of lowest order to that of any other mode can be quite large.

Finally, spatial coherency is examined by means of the mutual coherence function as defined in Ref. 2; it is shown that single mode oscillation is the necessary and sufficient condition for complete spatial coherency. Hence, in principle, a Fabry-Perot cavity with an active medium should generate spatial coherency.

In this analysis one makes the usual assumption of a linear medium, by which it is meant that the propagation vector **K** is a constant with respect to input, although the propagation vector may have a negative imaginary part which gives rise to gain in the system. In an actual system, however, **K** may be dependent to the input, constituting a case which is somewhat involved and beyond the scope of the present analysis.

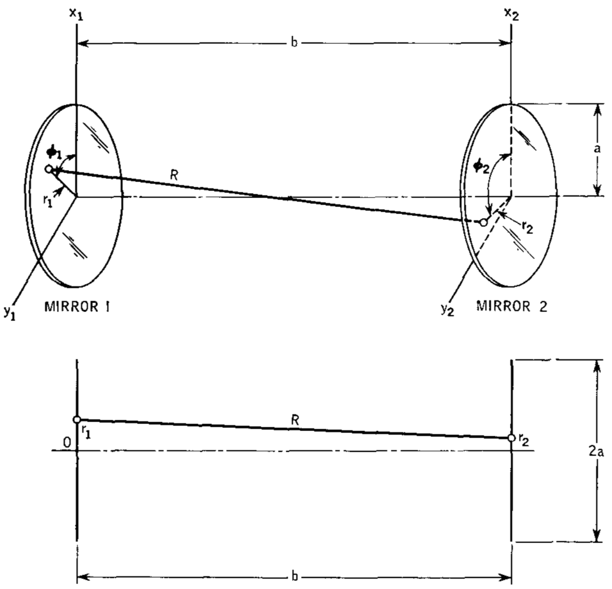
Analysis

Following the notation of Ref. 1, the integral equation governing the complex wave field distribution U can be written as

$$U_{q+1}(r_2, \phi_2) = \frac{j}{2\lambda} \int_0^a \int_0^{2\pi} U_q(r_1, \phi_1) \frac{e^{-KR}}{R} \left(1 + \frac{b}{R}\right) r_1 dr_1 d\phi_1, \tag{1a}$$

in which $U(r_1, \phi_1)$ is the field on the first mirror, K is the propagation constant, and other parameters are as indicated in Fig. 1. It can be shown that Eq. (1a) is a form of the Fresnel-Kirchhoff diffraction integral when the radius of curvature of the wave is sufficiently large. By separat-

Figure 1 **Arrangement of circular plane mirrors constituting the Fabry-Perot cavity.**



ing the dependences on γ and ϕ and carrying out the integration with respect to ϕ, the integral equation is reduced to

$$R_n'(r_2)\sqrt{r_2} = \lambda_2 \int_0^a K_n'(r_2, r_1)R_n(r_1)\,dr_1, \tag{1b}$$

where

$$K_n'(r_2, r_1) = \frac{j^{n+1}K}{b} J_n\left(K\frac{r_2 r_1}{b}\right)\sqrt{r_1 r_2}e^{-jK(r_1^2+r_2^2/2b)}. \tag{1c}$$

By an iterative procedure stationary fields were found as solutions in Ref. 1. The stationary fields are the proper functions of the integral equation, hence the normal modes of the cavity. The iterative procedure is as follows: With the integral equation in operator notation

$$U_{q+1} = LU_q, \tag{2}$$

where L is the integral operator, if we take a function U that can be expanded into a set of the proper function of L such that

$$U_0 = \sum_{i=1}^{k} C_i\phi_i,$$

then

$$U_n = L^n U_0 = \sum_{i=1}^{k} C_i\lambda_i^n\phi_i. \tag{3}$$

By working with this restricted class of U we avoid any assumption on the completeness of the proper function. Physically, it is clear that the ϕ_i's cannot be complete, for a wave which does not touch either of the mirrors cannot be expanded in ϕ_i's. But such waves are not pertinent for they disappear after a few iterations. Hence, to assume this restricted class of U does not occasion much loss of generality. Thus, when the largest proper value is non-degenerate so that the ordering $\lambda_1 > \lambda_2 \geq \lambda_3 \cdots \geq \lambda_K$ is possible, it is evident that the ratio $\lambda_j^n/\lambda_1^n \rightarrow 0$, as $n \rightarrow \infty$, for $J \neq 1$. The iterative procedure and the corresponding periodic structure thus give complete mode selection.

The case of the Fabry-Perot cavity is slightly more involved since because of multireflections one must deal with a sum of wave amplitudes. The wave amplitude incident on the mirror, $f(r)$, can be written as

$$f(r) = U_0 + K_1K_2U_2 + K_1^2K_2^2U_4 \cdots K_1^NK_2^NU_{2N}, \tag{4}$$

in which K_1, K_2 are the reflection matrix of the mirrors at either end of the cavity. It is possible to include the reflection loss in λ, however it is preferable to keep it in the present form since K_1, K_2 and λ are losses due to different physical phenomena.

Neglecting the effect of transit time on the coefficient C_1's, Eq. (4) can be rewritten as

$$f(r) = \sum_{n=0}^{N} (C_1\lambda_1^{2n}\phi_1 + C_2\lambda_2^{2n}\phi_2 + C_3\lambda_3^{2n}\phi_3 \cdots C_K\lambda_K^{2n}\phi_K)(K_1K_2)^n. \tag{5}$$

Physically, neglecting the effect of transit time means that either the incident wave has a sufficiently long coherence time or that the observation time is sufficiently long to make stationary the average value of the C_i's over the period of observation.

To consider the mode selection of the cavity one has to consider only the ratio of the coefficient of ϕ_i, M, to the largest coefficient M_1. Since

$$M_j = \sum_{n=1}^{N} C_j\lambda_j^{2n}(K_1K_2)^n = C_j\frac{1-(\lambda_j^2K_1K_2)^{N+1}}{1-\lambda_j^2K_1K_2},$$

$$\frac{M_j}{M_1} = \left[\frac{(K_1K_2\lambda_j^2)^{N+1}-1}{(K_1K_2\lambda_1^2)^{N+1}-1}\right]\left(\frac{C_j}{C_1}\right)\left(\frac{K_1K_2\lambda_1^2-1}{K_1K_2\lambda_j^2-1}\right). \tag{6}$$

It is convenient to consider two ranges of λ's, namely those for which $K_1K_2\lambda^2 > 1$ and $K_1K_2\lambda^2 < 1$. The first case is that in which the active medium has a gain greater than $(K_1K_2)^{-1/2}$ per pass, and the second case is that for the Fabry-Perot cavity without an active medium, or for a gain less than $(K_1K_2)^{-1/2}$ per pass. For the case $K_1K_2\lambda^2 < 1$ it is interesting to note that as N approaches ∞

$$\frac{M_j}{M_1} \rightarrow \frac{C_j}{C_1}\left(\frac{1-K_1K_2\lambda_1^2}{1-K_1K_2\lambda_j^2}\right). \tag{7}$$

Since this ratio is finite, all modes will be present at the ends, and complete mode selection is not possible. Hence, in theory the externally illuminated Fabry-Perot cavity will not in itself generate complete mode selection, although the amplitude of higher modes may be quite small in comparison with that of the lowest mode.

When an active medium is present such that $K_1K_2\lambda_2 > 1$, the situation is quite different, since as N approaches ∞, the ratio

$$\frac{M_j}{M_1} = T\left(\frac{\lambda_j}{\lambda_1}\right)^{2(N+1)} \rightarrow 0 \tag{8}$$

for all $j \neq 1$.

This case is very similar to that of the periodic structure. It is interesting to note that Eq. (8) implies that even if there are two modes having a gain per pass, the mode with the higher gain would dominate after a sufficient number of reflections.

We next consider the relation between mode selection and spatial coherency. It is well known[3] that the complex degree of coherence of the light on an aperture is defined as

$$\mu(R_1R_2) = \frac{J(R_1R_2)}{\sqrt{I(R_1)I(R_2)}}, \tag{9}$$

where $J(R_1, R_2)$ is the mutual intensity function, defined as $\langle U(R_1, t), U^*(R_2, t)\rangle$, and $I(R_1) = J(R_1, R_1)$ is the time average intensity at R_1. Any wave field $U(R, t)$ which can be expanded in the ϕ_i's can be put in the form

$$U_n(R, t) = \sum_{i=1}^{K} A_i(t)\phi_i(R). \tag{10}$$

If the original beam contains more than one mode, there will be more than one term in the series and the mutual intensity function will form a matrix with elements

$$J_{ij}(R_1R_2) = \langle A_i(t) A_j^*(t)\rangle\phi_i(R_1)\phi_j^*(R_2). \tag{11}$$

It is clear that in general

$$B_{ij} = \langle A_i(t) A_j^*(t)\rangle \tag{12}$$

is a Hermitian matrix with a positive definite quadratic form. From Eqs. (10), (11), and (12), Eq. (9) takes the form

$$\mu(R_1R_2) = \frac{\sum_{ij} B_{ij}\phi_i(R_1)\phi_j^*(R_2)}{[\sum_{ij} B_{ij}\phi_i(R_1)\phi_j^*(R_1) \sum_{ij} B_{ij}\phi_i(R_2)\phi_j^*(R_2)]^{1/2}}. \tag{13}$$

To show that $\mu(R_1R_2) < 1$ one considers the quantity

$$F(\alpha, \beta) = \sum_{ij} B_{ij}[\alpha\phi_i(R_1) + \beta\phi_i(R_2)] \cdot[\alpha\phi_j(R_1) + \beta\phi_j(R_2)]^*. \tag{14}$$

Since B_{ij} has a positive definite quadratic form for non-zero ϕ then, if ϕ_i and ϕ_j are not proportional, $F(\alpha, \beta) > 0$ for all non-zero α, β. Expanding Eq. (14) one obtains

$$F(\alpha, \beta) = a\,|\alpha|^2 + b\alpha\beta^* + b^*\alpha^*\beta + c\,|\beta|^2, \tag{15}$$

where

$$a = \sum_{ij} B_{ij}\phi_i(R_1)\phi_j^*(R_1),$$

$$b = \sum_{ij} B_{ij}\phi_i(R_1)\phi_j^*(R_2),$$

$$c = \sum_{ij} B_{ij}\phi_i(R_2)\phi_j^*(R_2).$$

If we set $\alpha = \sqrt{b^*/b}\,\theta$ and $\beta = \phi$, with θ and ϕ real, then Eq. (15) is reduced to

$$F(\alpha, \beta) = a\theta^2 + 2\sqrt{b^*b}\,\theta\phi + c\phi^2 > 0. \tag{16}$$

Hence $ac > bb^*$ or

$$\sum_{ij} B_{ij}\phi_i(R_1)\phi_j^*(R_1) \sum_{ij} B_{ij}\phi_i(R_2)\phi_j^*(R_2) > \left|\sum_{ij} B_{ij}\phi_i(R_1)\phi_j(R_2)\right|^2. \tag{17}$$

Thus the mutual coherence function is always less than unity and complete spatial coherency does not appear possible.[4]

When only one mode is present, Eq. (13) reduces to scalar form, from which it can be seen by inspection that the modulus of the mutual coherence function is unity, which implies complete spatial coherency.

Conclusion

This communication has considered the case of a Fabry-Perot cavity with light impinging on one end while the field distribution on the other end is observed. It has been shown that if one starts with a wave that can be expanded into the modes of the integral operator, and if the medium in the cavity is passive, more than one mode would be observed at the other end although the amplitude of the dominant mode can be much greater than that of the other modes. It was further shown that if there is present an active medium with sufficient gain there will be only a single mode, and the single mode is necessary and sufficient for complete spatial coherency.

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